CONGRUENCES MODULO PRIMES OF THE FORM $10n+1$

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ABSTRACT

The equation $x^n + y^n + z^n = 0$ had been shown to be impossible in $\mathbb{Z}_p\setminus\{0\}$. This result is known as Fermat’s Last Theorem. This paper shows that the above equation is also impossible in $\mathbb{Z}_p\mathbb{Z}_p\setminus\{0\}$, where $p$ is a prime of the form $p=10n+1$.

Keywords: prime, equation, congruence, modulo

INTRODUCTION

The end of the second thousand had witnessed the proof of Fermat’s Last Theorem (Edwards, 1977). This proof was searched by mathematicians during hundreds of years. Thousands of important articles were written in this subject. In this work we look for a generalization of Fermat’s Last Theorem to congruences modulo primes, that is, given a prime $p$, does the equation

$$x^n + y^n + z^n = 0$$

have a solution in $\mathbb{Z}_p\mathbb{Z}_p\setminus\{0\}$, for every $n$? The answer is positive if $p=3$, for take $x=y=z=1$. But if $p\neq 3$, the answer seems to be negative, for if $p=2^s n+1$ is prime with $1 \leq s \leq 4$ and $n > 2$, then the equation $x^n + y^n + z^n = 0$ has no solution in $\mathbb{Z}_p\mathbb{Z}_p\setminus\{0\}$, by Theorem A’ (Cheaito, 2001). In other words if $p=2^s n+1$ is prime with $1 \leq s \leq 4$, $n > 2$ and if $x$, $y$ and $z$ are integers such that $x^n + y^n + z^n \equiv 0 \pmod{p}$, then $p$ divides one of $x$, $y$ and $z$.

Thus as for $n=5$, we have that $2n+1=11$ and $2^3 n+1=41$ are prime, we get that the equation $x^5 + y^5 + z^5 = 0$ is impossible in $\mathbb{Z}/11\mathbb{Z}\setminus\{0\}$ and in $\mathbb{Z}/41\mathbb{Z}\setminus\{0\}$, that is if $x$, $y$ and $z$ are integers such that $x^5 + y^5 + z^5 \equiv 0 \pmod{11}$, then 11 divides one of $x$, $y$ and $z$ and also if $x^5 + y^5 + z^5 \equiv 0 \pmod{41}$, then 41 divides one of $x$, $y$ and $z$.

In this paper we shall show

**Theorem A:** Let $n$ be an odd number $\geq 5$. If $p=10n+1$ is prime and if $x$, $y$ and $z$ are integers such that $x^n + y^n + z^n \equiv 0 \pmod{p}$, then $p$ divides one of $x$, $y$ and $z$. □
As a corollary to Theorem A we state the following

**Corollary 1:** If \( n \) is an odd number \( \geq 5 \), and if \( p=10n+1 \) is prime, then the equation

\[
x^n + y^n + z^n = 0
\]

is impossible in \( \mathbb{Z}_p \setminus \{0\}. \]

The proof of Theorem A is an immediate consequence of the following theorem:

**Theorem 1:** Let \( n \) be an odd number \( \geq 5 \). If \( p=10n+1 \) is prime and

\[
u + v + w \equiv 0 \pmod{(p)} \quad \text{and} \quad u^{10} = v^{10} = w^{10} \pmod{(p)}
\]

then \( p \) divides one of \( u, v \) and \( w \).

**Proof of Theorem 1:**

All congruences are modulo \( p \).

Assume that \( p \) does not divide \( uvw \). If \( u \equiv v \), then we get \( 2u \equiv -w \), and so

\[
2^5 = \pm 1
\]

hence either \( 33 \equiv 0 \) or \( 31 \equiv 0 \). If \( 33 \equiv 0 \), then \( 10n+1=3 \) or \( 10n+1=11 \), which gives that \( 10n=2 \) or \( 10n=10 \), which is impossible. If \( 31 \equiv 0 \), then \( 10n+1=31 \), and so \( n=3 \), impossible. Therefore \( u \equiv v \) and \( u \equiv w \) are impossible.

We have \( u^{10} = v^{10} = w^{10} \), hence and because of symmetry, we get

\[
u^5 = v^5 = w^5 \quad \text{or} \quad u^5 = v^5 = -w^5.
\]

Suppose that

\[
u^5 = v^5 = w^5.
\]

Then \( u^5 - w^5 = 0 \) and \( v^5 - w^5 = 0 \). But

\[
u^5 - w^5 = (u-w)(u^4 + u^3 w + u^2 w^2 + uw^3 + w^4)
\]

and

\[
v^5 - w^5 = (v-w)(v^4 + v^3 w + v^2 w^2 + vw^3 + w^4)
\]

and so as \( u = w \) and \( v = w \) are impossible, then

\[
u^4 + u^3 w + u^2 w^2 + uw^3 + w^4 = 0
\]

and

\[
v^4 + v^3 w + v^2 w^2 + vw^3 + w^4 = 0.
\]

This implies

\[
(u^4 - v^4) + (u^3 - v^3)w + (u^2 - v^2)w^2 + (u-v)w^3 = 0
\]

hence

\[
(u-v)(u+v)(u^2 + v^2) + (u-v)(u^2 + v^2 + uv)w + (u-v)(u+v)w^2 + (u-v)w^3 = 0
\]

But \( u+v = -w \) and \( u = v \) is impossible, hence
whence 

$\text{uvw} = 0$

which is impossible.

Assume that 

$u^5 = v^5 = -w^5$.

Then 

$u^5 + w^5 = 0$ and $v^5 + w^5 = 0$.

But 

$u^5 + w^5 = (u+w)(u^4 - u^3 w + u^2 w^2 - uw^3 + w^4)$

and 

$v^5 + w^5 = (v+w)(v^4 - v^3 w + v^2 w^2 - vw^3 + w^4)$

and so as $u = -w$ and $v = -w$ are impossible, then 

$u^4 - u^3 w + u^2 w^2 - uw^3 + w^4 = 0$

and 

$v^4 - v^3 w + v^2 w^2 - vw^3 + w^4 = 0$.

This implies that 

$(u^4 - v^4)w - (u^2 - v^2)w^2 - (u-v)w^3 = 0$

hence 

$(u-v)(u+v)(u^2 + v^2) - (u-v)(u^2 + v^2)(u^2 + v^2) - (u-v)(u^2 + v^2)(u^2 + v^2) = 0$

But $u+v = -w$ and $u = v$ is impossible, hence 

$-2(u^2 + v^2 - uv)w - 2w^2 = 0$

whence 

$-2(u^2 + v^2 - uv)w - 2w^2 = 0$

which gives 

$2(u^2 + v^2) + uv + 2w^2 = 0$

hence as $u^2 + v^2 = (u+v)^2 - 2uv = w^2 - 2uv$, then 

$2(w^2 - 2uv) + uv + 2w^2 = 0$

and so 

$4w^2 - 3uv = 0$

whence 

$4w^2 = 3uv$.

On the other hand we have $u^5 - v^5 = 0$ and $u = v$ is impossible, hence 

$u^4 + u^3 v + u^2 v^2 + uv^3 + v^4 = 0$

and so 

$u^4 + v^4 + uv(u^2 + v^2) + u^2 v^2 = 0$.

However 

$u^2 + v^2 = (u+v)^2 - 2uv = w^2 - 2uv$. 
Therefore
\[ u^4 + v^4 = (u^2 + v^2)^2 - 2u^2 v^2 = (w^2 - 2uv)^2 - 2u^2 v^2 = w^4 - 4uvw^2 + 2u^2 v^2 \]
and so
\[ w^4 - 4uvw^2 + 2u^2 v^2 + uv(w^2 - 2uv) + u^2 v^2 = 0 \]
which gives
\[ w^4 - 3uvw^2 + u^2 v^2 = 0 \]
hence
\[ w^2 (w^2 - 3uv) + u^2 v^2 = 0 \]
and so as \( 4w^2 = 3uv \), then
\[ w^2 (w^2 - 4w^2) + u^2 v^2 = 0 \]
whence
\[ -3w^4 + u^2 v^2 = 0 \]
and so
\[ 9(-3w^4 + u^2 v^2) = 0 \]
which gives \(-27w^4 + (3uv)^2 = 0\), and hence \(-27w^4 + (4w^2)^2 = 0\), whence \(-11w^4 = 0\), which yields that \( w^4 = 0 \), impossible. Thus \( p \) divides \( uvw \).

Proof of Theorem A:

Take \( u = x^n \), \( v = y^n \) and \( w = z^n \). Then
\[ u + v + w = 0 \pmod{p}. \]

Since \( p \) is prime, \( \mathbb{Z}/p\mathbb{Z} \setminus \{0\} \) is a finite group of order \( p-1 \). But as \( p-1=10n \), then \( \alpha^{10n} = 1 \), for every \( \alpha \) in \( \mathbb{Z}/p\mathbb{Z} \setminus \{0\} \), and so
\[ u^{10} = v^{10} = w^{10} = 1 \]
whence \( u^{10} = v^{10} = w^{10} \pmod{p} \). It follows from Theorem 1 that \( p \) divides \( uvw \), and so \( p \) divides \( (xyz)^n \). As \( p \) is prime, then \( p \) divides \( xyz \), and so it divides one of \( x \), \( y \), and \( z \). \( \square \)

REFERENCES
