ON THE INVERSE LIMIT OF FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRAS

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ABSTRACT

This study proves that every finite dimensional homomorphic image of a prosemisimple Lie algebra $L = \varprojlim L_n$ $(n \in N)$ is also semisimple in the case when the dimensions of all simple factor ideals of all L_n are bounded.

Keywords: semisimple Lie algebras, Lie groups

INTRODUCTION

Most of the general theory on Lie algebras has been established for finite dimensional Lie algebras (Bourbaki, 1989; Hochschild, 1981). In fact very little is known about the general theory of infinite dimensional Lie algebras.

An important class of such Lie algebras are the profinite dimensional Lie algebras $L = \varprojlim L_i$ which are inverse limits of finite dimensional Lie algebras. Such Lie algebras appear as the Lie algebras of proaffine algebraic groups which play an important role in the representation theory of Lie groups (Nahlus, 1994; 2002).

In this paper, it is proven that every finite dimensional homomorphic image of a prosemisimple Lie algebra $L = \varprojlim L_n$ $(n \in N)$ where each L_n is a finite dimensional semisimple Lie algebra is also semisimple if the dimensions of all simple factor ideals of all L_n are bounded.

PRELIMINARIES

All Lie algebras in this paper are considered over a fixed algebraically closed field K of characteristic 0.

Definition 1. *Let I be a set with a partial ordering ≤. Suppose I is directed upwards, i.e., for every $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. For every $i \in I$, let S_i be a set such that for every pair $(i,j) \in I \times I$ with $j \geq i$, there is a map $\pi_{ji}: S_j \to S_i$ satisfying the following two conditions :*

(i) *π_{ii} is the identity map for every $i \in I$; and*

(ii) *if $i \leq j \leq k$, then $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$,* *then* $(S_i, \pi_{ji})_{ij\in I}$ *is called an inverse system and the maps* π_{ji} *:* $S_j \rightarrow S_i$ *are called the transition maps of the inverse system. The inverse limit of this system, denoted by* $\varprojlim S_i$*, is the subset of the Cartesian product* $\prod_{i\in I} S_i$ *consisting of all elements* $s = (s_i)_{i\in I}$ *such that* $\pi_{ji}(s_j)=s_i$ *for every* $j\geq i$*. In other words,* $\varprojlim S_i$ *is the set of all families of elements* $(s_i)_i \in I$ *which are compatible with the transition maps* $\pi_{ji} : S_j \rightarrow S_i$*.*

If (S_i) is an inverse system, let $\pi_i : \varprojlim S_i \rightarrow S_i$ be the canonical projection sending (s_i) to s_i. Then $\pi_i = \pi_{ji} \circ \pi_j$ for every $i \leq j$. A surjective inverse system is an inverse system whose transition maps are surjective.

For the convenience of the reader some basic facts are recalled about finite dimensional semisimple Lie algebras.

Theorem 1. (Humphreys, 1972). *Let L be a finite dimensional semisimple Lie algebra over K.*

1. There exist simple ideals $L_1, L_2 ..., L_t$ [unique up to order] such that $L = L_1 \oplus .. \oplus L_t$. Moreover, each ideal of L is a sum of such simple ideals.
2. All ideals and homomorphic images of L are semisimple.
3. There exist sub-Lie algebras $L_1, L_2, .., L_n$ of L, each isomorphic to S1(2,K), such that $L = L_1 + .. + L_n$.

PROSEMISIMPLE LIE ALGEBRAS

Proposition 1. Let $L = \varprojlim L_i$ $i \in I$, where I is a directed upwards set with a partial ordering $\leq$, be the inverse limit of a surjective inverse system of finite dimensional semisimple Lie algebras over K. Then, $L \cong \prod_{w\in\Lambda} S_w$, where the S_w are isomorphic to simple finite dimensional Lie algebras.

Proof. Let L_i be the ith projection of L. Since L_i is semisimple, it follows from Theorem 1 that $L_i = \bigoplus_{t=1}^{p} S_t^i$, $p \in N$, where the S_t^i are simple ideals of L_i. Similarly $L_j = \bigoplus_{r=1}^{m} S_r^j$, $m \in N$ and so on. It is first noted that if $j \geq i$, then for a simple ideal S_t^i in L_i there exists a unique S_r^j in L_j such that $\pi_{ji}(S_r^j) = S_t^i$ [This is due to the fact that the preimage of S_t^i is an ideal of L_j, and thus it is a sum of some simple S_r^j. For simplicity, take m = 2, and $\pi_{ji}(S_1^j \oplus S_2^j) = S_t^i$. Let π'_{ji} be the restriction of π_{ji} to

$S_1^j \oplus S_2^j$. Then, Kernel π'_{ji}, which is an ideal of $S_1^j \oplus S_2^j$ is either 0, which is not the case since π'_{ji} cannot be injective, or $S_1^j \oplus S_2^j$ which is also not the case since $S_t^i \neq 0$. Thus we are left with the case that Kernel π'_{ji} is either S_1^j or S_2^j. Hence, there exists only one S_r^j such that $\pi'_{ji}\left(S_r^j\right) = S_t^i$]. Let π''_{ji} be the restriction of π_{ji} to S_r^j. If $\pi''_{ji}\left(S_r^j\right) = S_t^i$, and $\pi''_{kj}\ (S_h^k) = S_r^j$, k $\geq$ j $\geq$ i, then,

$\pi''_{ki}\ (S_h^k) = \pi_{ki}\,(S_h^k) = (\pi_{ji} \text{ o } \pi_{kj})\, S_h^k = \pi_{ji}\ (S_r^j) = S_t^i$.

Thus $\pi''_{ki} = \pi''_{ji} o \pi''_{kj}$. Therefore, the S_t^i form a surjective subinverse system with transition maps π''_{ji}, the restriction of π_{ji} to S_r^j. Define now a vertical line to be a surjective subinverse system $\{S_t^i, \pi''_{ji}\}$. Since Kernel π''_{ji} is an ideal of S_r^j, Kernel π''_{ji} is either 0 or S_r^j. If it is S_r^j, then S_t^i = 0; otherwise π''_{ji} is injective. But it is also surjective. Thus π''_{ji} is an isomorphism and thus the inverse limit of the inverse system $\{S_t^i, \pi''_{ji}\}$ is isomorphic to S_t^i. Let $\{S_w, w \in \Lambda\}$ be the collection of the inverse limits of all possible vertical lines. Then, $L \cong \prod_{w\in\Lambda} S_w$. Let l $\in L$, l = (l_i) and l_i can be written uniquely as $\sum_{t=1}^{p} s_t^i$, thus $l = \left(\sum_{t=1}^{p} s_t^i\right)$. The s_t^i form a compatible inverse system. This is due to the fact that the l_i are compatible. Let $\hat{\pi}_{ji}, \hat{\pi}_{kj}$ be the restrictions of π_{ji} and π_{kj} to s_t^i and s_h^k respectively. If $\hat{\pi}_{ji}\left(s_r^j\right) = s_t^i$ and $\hat{\pi}_{ki}(s_h^k) = s_r^j$, then, since $\pi_{ki}(l_k) = l_i$, we get $\pi_{ki}\left(\sum s_h^k\right) = l_i$,i.e. $\sum \pi_{Ki}(s_h^k) = l_i$. But l_i is written uniquely as $\sum^{p} s_t^i$. Thus, $\hat{\pi}_{ki}\left(s_h^k\right) = s_t^i$. Let f: L→ $\prod_{w\varepsilon\Lambda} S_w$ be the Lie algebra homomorphism given by : $f((l_i)) = f\left(\left(\sum_{t=1}^{p} s_t^i\right)\right) = \prod_{w\in\Lambda} s_w^{t,i}$, where $\{S_w^{t,i}\}$ is the collection of inverse limits of inverse systems (s_t^i). f is well defined. Moreover, if $(l_i) = \left(\sum_{t=1}^{p} s_t^i\right) \in$ Ker f, then, f((l_i))= (0), *i.e.*

$\prod_{w\in\Lambda} s_w^{t,i}$ =(0). Hence each S_t^i = 0 for all t and for all i. Thus (l_i) = 0 and f is injective.

In addition, let s = $\prod_{w\in\Lambda} s_w^{t,i}$ be an element of $\prod_{w\varepsilon\Lambda} S_w$, and let $l = (l_i)$ where l_i is the finite sum of the the ith projection of the different $S_w^{t,i}$. Then f(l)=s and f is surjective. Therefore, f is a Lie algebra isomomorphism and $L \cong \prod_{w\varepsilon\Lambda} S_w$.

FINITE CODIMENSIONAL IDEALS OF PROSEMISIMPLE LIE ALGEBRAS

Lemma 1. *Let* $L = \prod L_i$, $i \in N$, *where the* L_i *are finite dimensional simple Lie algebras over K. If A is an ideal of L of finite codimension, then A contains all but a finite number of the* L_i.

Proof. The proof will be done by contradiction. Suppose we have an infinite sequence L_1, L_2, L_3..of the L_i not in A. For an arbitrary L_i of the L_i in our sequence, A $\cap L_i$ is an ideal of L_i. Thus A $\cap$ L_i ={0} or A $\cap L_i = L_i$ which is not the case since L_i is not in A. Thus, A $\cap$ L_i = {0}. Let d = dim L/A and let $V = L_1 \oplus L_2 \oplus .. \oplus L_t$ be such that dim $V > d$. Let $\pi' : V \to L/A$ be the restriction of the canonical projection $L \to L/A$ to V. Ker π' is an ideal of V; hence, Ker π' is a direct sum of some L_i in V, say, Ker $\pi' = L_1 \oplus L_2 \oplus .. \oplus L_m$, $1 \le m \le t$. Then,

$$\pi'(L_1) = \pi'(L_2) = ... = \pi'(L_m) = 0.$$

Thus, $L_1, L_2,.., L_m$ would all be included in A, which is not the case. Hence, Ker π' = {0} and consequently we get a contradiction, since dim $V >$ dim L/A. Thus, A contains all but a finite number of the L_i.

Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & \_1 \end{pmatrix}$ be the standard basis of S1(2,K). Then, [x,y]= h, [h,x]=2x and [h,y]=-2y.

Lemma 2. *Let* $L = \prod L_i$, $i \in N$, *where the* L_i *are finite dimensional simple Lie algebras over K. Suppose that the dimensions of all the* L_i *are bounded and let A be an ideal of L that contains the elements* $\overline{x_1} = (k_0^1 x_1, k_1^1 x_1,...), \overline{x_2} = (k_0^2 x_2, k_1^2 x_2,...),.., \overline{x_t} = (k_0^t x_1, k_1^t x_t,...)$ *where the* k_i^j *are non-zero constants, then A = L.*

Proof. According to Theorem 1, each L_i is of the form $L_i^1 + L_i^2 + ..L_i^{t_i}$ where each $L_i^{t_i}$ is a sub-Lie algebra of L_i isomorphic to S1(2,K). Since the dimensions of all the L_i are bounded, we can assume without loss of generality that all the t_i are equal to a certain integer t.

For an arbitrary element l *in* L,

$$l=(l_0^1+l_0^2+..+l_0^1;l_1^1+l_1^2+..+l_1^t;l_2^1+l_2^2+..+l_2^t;..)$$
$$=(l_0^1,l_1^1,l_2^1,,..)+(l_0^2,l_1^2,l_2^2,..)+..+(l_0^t,l_1^t,l_2^t,..).$$

Thus,

$$l=(\alpha_0^1x_1+\beta_0^1y_1+\gamma_0^1h_1;\alpha_1^1x_1+\beta_1^1y_1+\gamma_1^1h_1;..)$$
$$+(\alpha_0^2x_2+\beta_0^1y_2+\gamma_0^2h_2;\alpha_1^2x_2+\beta_1^1y_2+\gamma_1^2h_2;..)+..$$
$$+(\alpha_0^tx_t+\beta_0^ty_t+\gamma_0^th_t;\alpha_1^tx_tt+\beta_1^ty_t+\gamma_1^th_t;..),$$ where the α_i^j,β_i^j and γ_i^j are constants.

For arbitrary integer n, $1\le n\le t$, let $\bar{h}_n=\left(\frac{\alpha_0^n}{2k_0^n}h_n;\frac{\alpha_1^n}{2k_1^n}h_n;...\right)\in L.$

Since A is an ideal of L, $\left[\bar{h}_n,\bar{x}_n\right]=\left(\alpha_0^nx_n,\alpha_1^nx_n;...\right)\in$ A. Let $\bar{y}_n=\left(\frac{\gamma_0^n}{k_0^n}y_n;\frac{\gamma_1^n}{k_1^n}y_n;...\right)\in L.$ Then, $\left[\bar{x}_n,\bar{y}_n\right]=\left(\gamma_0^nh_n,\gamma_1^nh_n;...\right)\in$ A. Let $\bar{y}'_n=\left(\frac{\beta_0^n}{2k_0^n}y_n;\frac{\beta_1^n}{2k_1^n}y_n;...\right)\in L.$

Then $\left[\bar{x}_n,\bar{y}'_n\right]=\left(\frac{\beta_0^n}{2}h_n,\frac{\beta_1^n}{2}h_n;...\right)\in$ A.

Thus, $\left[\left(y_n,y_n;..\right),\left(\frac{\beta_0^n}{2}h_n;\frac{\beta_1^n}{2}h_n;...\right)\right]=\left(\beta_0^nh_n,\beta_1^nh_n;...\right)\in$ A.

Hence A contains

$$\left(\alpha_0^nx_n;\alpha_1^nx_n;...\right)+\left(\beta_0^ny_n;\beta_1^ny_n;...\right)+\left(\gamma_0^nh_n;\gamma_1^nh_n;...\right)=(\alpha_0^nx_n+\beta_0^ny_n+\gamma_0^nh_n;\alpha_1^nx_n+\beta_1^ny_n+\gamma_1^nh_n;..)$$

This being true for all $1\le n\le t$, we conclude that

$l=(l_0^1,l_1^1,l_2^1,..)+(l_0^2,l_1^2,l_2^2,..)+..+(l_0^t,l_1^t,l_2^t,..)\in$ A, and A=L.

Theorem 2. Let L = $\prod L_i$, $i\in N$, where the L_i are finite dimensional simple Lie algebras over K. Suppose that the dimensions of all the L_i are bounded and let A be an ideal of L of finite codimension then L/A is semi simple.

Proof. As mentioned previously, each L_i is of the form $L_i^1+L_i^2+..L_i^{t_i}$ where each $L_i^{t_i}$ is a sub-Lie algebra of L_i isomorphic to S1(2,K). We can assume without loss of generality that all the t_i are equal to a certain integer t.

We will first prove that if J is a confinite dimensional ideal of L and J contains every L_i, then J = L.

This is done as follows: Let n = dim L/ J. Since J contains every L_i, J contains $\bigoplus L_i$. For every integer j,1≤j≤t, let S_j be the subspace of L spanned by the vectors { (x_j; x_j; x_j;….); ($2x_j$; $3x_j$;…); ($2^2 x_j$; $3^2 x_j$;…), .. ($2^n x_j$; $3^n x_j$;…)}.

These vectors are linearly independent:

If $\alpha_0 (x_j;x_j,\ldots) + \alpha_1(2x_j; 3x_j;\ldots.) +\ldots+ \alpha_n(2^n x_j; 3^n x_j;\ldots)= \vec{0}$. Then,

$\alpha_0 x_j+2\alpha_1 x_j+\ldots+ 2^n \alpha_n x_j=0;$

$\alpha_0 x_j+3\alpha_1 x_j+\ldots+ 3^n \alpha_n x_j=0;\ldots;$

$\alpha_0 x_j+(n+2) \alpha_1 x_j+\ldots+(n+2)^n\alpha_n x_j=0.$

$$i.e. \begin{pmatrix} 1 & 2 & 2^n \\ 1 & 3 & 3 \\ & & \\ 1 & (n+2) & (n+2)^n \end{pmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}.$$

But the first matrix is a vandermonde matrix so it is invertible and hence $\alpha_0 = \alpha_1 = \ldots= \alpha_n = 0$. Thus, the vectors are linearly independent and they form a basis of S_j. Thus $\dim S_j=n+1$. But dim L / J =n, which implies that $J \cap S_j$ is non- zero.

Let a be a non- zero element of the intersection of J and Sj. Then,

$a= \gamma_0 (x_j;x_j;..)+ \gamma_1 (2x_j;3x_j;..)+..+ \gamma_n(2^n x_j;3^n x_j;..)$

$=(\gamma_0 x_j+2\gamma_1 x_j+\ldots+2^n\gamma_n x_j; \gamma_o x_j+3\gamma_1 x_j+\ldots+3^n\gamma_n x_j;\ldots)$. Thus,

$a=(\beta_o x_j ; \beta_1 x_j ;\ldots; \beta_{n+2}x_j;\ldots)=(\beta_m x_j)_{m\in N}$, where

$\beta_o= \gamma_0+ 2\gamma_1+\ldots +2^n \gamma_n ;$

$\beta_1= \gamma_0+ 3\gamma_1+\ldots +3^n \gamma_n ;\ldots;$

$\beta_{n-1}= \gamma_0+(n+1)\gamma_1+\ldots +(n+1)^n \gamma_n ;\ldots$

Now, if there exist (n+1) or more β_m which are zeros, then, assume without loss of generality that $\beta_0= \beta_1= \beta_2 = ..=\beta_n=0$. Then,

$\gamma_0+ 2\gamma_1+..2^n\gamma_n=0;\ldots; \gamma_0+(n+2) \gamma_1+\ldots+(n+2)^n\gamma_n=0.$

$$i.e. \begin{pmatrix} 1 & 2 & 2^n \\ 1 & 3 & 3 \\ & & \\ 1 & (n+2) & (n+2)^n \end{pmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \\ \gamma_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

But the first matrix is a vandermonde matrix which is invertible, thus $\gamma_0=\gamma_1=\ldots=\gamma_n= 0$ and a=0, contradicting the assumption that a $\neq$ 0. Therefore, $\beta_m =0$ for at most n β_m . Let k be the number of entries where $\beta_m =0$, 1≤ k≤ n, we can assume without loss of generality that the only entries of a for which $\beta_m =0$ are the first *k* entries. Since *J* contains $\bigoplus L_i$, J contains the element $a'=(x_j,x_j,....,x_j,0,0,...)$, *where* only k entries are x_j and the rest are zeros. Thus J contains the element $a+a'=(x_j,x_j,x_j...., \beta_{k+1}x_j,...)$. Since this is true for all $1 \le j \le t$, it follows by Lemma 2 that $J = L$.

Now, back to our problem. A is a cofinite dimensional ideal of L. By Lemma 1, A contains all but a finite number of the L_i, say, all but p-ones of L_i. Let V be the direct sum of these p-ones. Then A $\bigoplus V$ contains all the L_i, and it is of finite codimension. Thus, by what we proved at the begining, A $\bigoplus V = L$. L/A = (A $\bigoplus V)/A \cong V$. Hence, L/A is a product of a finite number of copies of L_i.

Corollary 1. *Let* $L=\varprojlim L_i$, $i \in N$, *where each* L_i *is a finite dimensional semisimple Lie algebra. Suppose that the dimensions of all simple factor ideals of all* L_i *are bounded. Then every finite dimensional homomorphic image of L is semisimple.*

Proof. The proof follows directly by combining Proposition 1 and Theorem 2.

Corollary 2. *Let* $L = \prod L_i$, $i \in N$, *where the* L_i *are finite dimensional simple Lie algebras over K. Suppose that the dimensions of all the* L_i *are bounded, and let W be a finite dimensional Lie algebra over K. Suppose* $f: L \to W$ *is a Lie algebra homomorphism such that* $f(L_i) = 0$ *for every* $L_i \subset L$. *Then* $f\left(\prod L_i\right) = 0$.

Proof. Ker f is an ideal of L. L / Ker f $\cong$ Image $f \subset W$, but W is of finite dimension. Thus Ker f is a finite codimensional ideal of L. Moreover, Ker f contains every $L_i \subset L$. Hence, by first part of Theorem 2, Ker f = $\prod L_i$, and f= 0.

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