ON THE INVERSE LIMIT OF FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRAS

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ABSTRACT
This study proves that every finite dimensional homomorphic image of a prosemisimple Lie algebra \( L = \lim_{n \in \mathbb{N}} L_n \) is also semisimple in the case when the dimensions of all simple factor ideals of all \( L_n \) are bounded.

Keywords: semisimple Lie algebras, Lie groups

INTRODUCTION

Most of the general theory on Lie algebras has been established for finite dimensional Lie algebras (Bourbaki, 1989; Hochschild, 1981). In fact very little is known about the general theory of infinite dimensional Lie algebras.

An important class of such Lie algebras are the profinite dimensional Lie algebras \( L = \lim_{i \in I} L_i \) which are inverse limits of finite dimensional Lie algebras. Such Lie algebras appear as the Lie algebras of proaffine algebraic groups which play an important role in the representation theory of Lie groups (Nahlus, 1994; 2002).

In this paper, it is proven that every finite dimensional homomorphic image of a prosemisimple Lie algebra \( L = \lim_{n \in \mathbb{N}} L_n \) where each \( L_n \) is a finite dimensional semisimple Lie algebra is also semisimple if the dimensions of all simple factor ideals of all \( L_n \) are bounded.

PRELIMINARIES

All Lie algebras in this paper are considered over a fixed algebraically closed field \( K \) of characteristic 0.

**Definition 1.** Let \( I \) be a set with a partial ordering \( \leq \). Suppose \( I \) is directed upwards, i.e., for every \( i, j \in I \) there exists \( k \in I \) such that \( i \leq k \) and \( j \leq k \). For every \( i \in I \), let \( S_i \) be a set such that for every pair \( (i, j) \in I \times I \) with \( j \geq i \), there is a map \( \pi: S_j \rightarrow S_i \) satisfying the following two conditions:
(i) \( \pi_i \) is the identity map for every \( i \in I \); and
(ii) if \( i \leq j \leq k \), then \( \pi_k = \pi_j \circ \pi_i \).
then \((S_i, \pi_{ji})_{i,j}\) is called an inverse system and the maps \(\pi_{ji}\) : \(S_j \to S_i\) are called the transition maps of the inverse system. The inverse limit of this system, denoted by \(\text{lim}_{i\in I} S_i\), is the subset of the Cartesian product \(\prod_{i\in I} S_i\) consisting of all elements \(s = (s_i)_{i\in I}\) such that \(\pi_{ji}(s_j) = s_i\) for every \(j \geq i\). In other words, \(\text{lim}_{i\in I} S_i\) is the set of all families of elements \((s_i)_{i\in I}\) which are compatible with the transition maps \(\pi_{ji} : S_j \to S_i\).

If \((S_i)\) is an inverse system, let \(\pi_i : \text{lim}_{i\in I} S_i \to S_i\) be the canonical projection sending \((s_i)\) to \(s_i\). Then \(\pi_i = \pi_{ji} \circ \pi_j\) for every \(i \leq j\). A surjective inverse system is an inverse system whose transition maps are surjective.

For the convenience of the reader some basic facts are recalled about finite dimensional semisimple Lie algebras.

**Theorem 1.** (Humphreys, 1972). Let \(L\) be a finite dimensional semisimple Lie algebra over \(K\).

1. There exist simple ideals \(L_1, L_2, \ldots, L_t\) [unique up to order] such that \(L = L_1 \oplus \cdots \oplus L_t\). Moreover, each ideal of \(L\) is a sum of such simple ideals.
2. All ideals and homomorphic images of \(L\) are semisimple.
3. There exist sub-Lie algebras \(L_1, L_2, \ldots, L_n\) of \(L\), each isomorphic to \(S(2, K)\), such that \(L = L_1 + \cdots + L_n\).

**PROSEMISIMPLE LIE ALGEBRAS**

**Proposition 1.** Let \(L = \text{lim}_{i\in I} L_i\) be the inverse limit of a surjective inverse system of finite dimensional semisimple Lie algebras over \(K\). Then, \(L \cong \bigoplus_{\Lambda \in I} S_{\Lambda}\), where the \(S_{\Lambda}\) are isomorphic to simple finite dimensional Lie algebras.

**Proof.** Let \(L_i\) be the \(i\)th projection of \(L\). Since \(L_i\) is semisimple, it follows from Theorem 1 that \(L_i = \bigoplus_{\rho=1}^{p} S_{\rho i}\), \(p \in \mathbb{N}\), where the \(S_{\rho i}\) are simple ideals of \(L_i\).

Similarly \(L_j = \bigoplus_{r=1}^{m} S_{r j}\), \(m \in \mathbb{N}\) and so on. It is first noted that if \(j \geq i\), then for a simple ideal \(S_{i j}\) in \(L_i\) there exists a unique \(S_{r j}\) in \(L_j\) such that \(\pi_{ji}(S_{r j}) = S_{i j}\). This is due to the fact that the preimage of \(S_{i j}\) is an ideal of \(L_j\), and thus it is a sum of some simple \(S_{r j}\). For simplicity, take \(m = 2\), and \(\pi_{ji}(S_{1 j} \oplus S_{2 j}) = S_{i j}\). Let \(\pi_{ji}'\) be the restriction of \(\pi_{ji}\) to
Then, Kernel $\pi'_{ji}$, which is an ideal of $S'_1 \oplus S'_2$, is either 0, which is not the case since $\pi'_{ji}$ cannot be injective, or $S'_1 \oplus S'_2$ which is also not the case since $S'_i \neq 0$.

Thus we are left with the case that Kernel $\pi'_{ji}$ is either 0, which is not the case since $\pi'_{ji}$ cannot be injective, or $S'_1 \oplus S'_2$ which is also not the case since $0 \neq S'_i$.

Thus we are left with the case that Kernel $\pi'_{ji}$ is either $S'_j$ or $S'_k$. Hence, there exists only one $S'_r$ such that $\pi'_{ji} (S'_r) = S'_r$. Let $\pi''_{ji}$ be the restriction of $\pi_{ji}$ to $S'_r$. If $\pi''_{ji} (S'_r) = S'_r$, and $\pi'_{ij}$ ($S^k_h = S'_r$, $k \geq j \geq i$, then,

$$\pi''_{kl} (S^k_h) = \pi_{kl} (S^k_h) = (\pi_{ji} \circ \pi_{kj}) (S^k_h) = \pi_{ji} (S'_r) = S'_r.$$ 

Thus $\pi''_{kl} = \pi_{kl} \circ \pi'_{ij}$. Therefore, the $S'_r$ form a surjective subinverse system with transition maps $\pi''_{ji}$, the restriction of $\pi_{ji}$ to $S'_r$. Define now a vertical line to be a surjective subinverse system $\{S'_r, S''_{ji}\}$. Since Kernel $\pi''_{ji}$ is an ideal of $S'_r$, Kernel $\pi'_{ji}$ is either 0 or $S'_r$. If it is $S'_r$, then $S'_r = 0$; otherwise $\pi'_{ji}$ is injective. But it is also surjective. Thus $\pi''_{ji}$ is an isomorphism and thus the inverse limit of the inverse system

$\{S'_r, \pi''_{ji}\}$ is isomorphic to $S'_r$. Let $\{S_w, w \in W\}$ be the collection of the inverse limits of all possible vertical lines. Then, $L \equiv \prod_{w \in W} S_w$. Let $l \in L$, $l = (l_i)$ and $l_i$ can be written uniquely as $\sum_{i=1}^{p} l'_i$, thus $1 = \left( \sum_{i=1}^{p} S'_i \right)$. The $S'_i$ form a compatible inverse system. This is due to the fact that the $l_i$ are compatible. Let $\tilde{\pi}_{ji}$, $\tilde{\pi}_{kj}$ be the restrictions of $\pi_{ji}$ and $\pi_{kj}$ to $S'_i$ and $S'_k$ respectively. If $\tilde{\pi}_{ji} (S'_i) = S'_i$ and $\tilde{\pi}_{kj} (S'_k) = S'_k$, then, since $\pi_{kl} (l_k) = l_i$, we get $\pi_{kl} (\sum_{h} S'_h) = l_i$. But $l_i$ is written uniquely as $\sum_{h} S'_h$. Thus,

$$\tilde{\pi}_{ki} (S'_h) = l'_i.$$ 

Let $f: L \rightarrow \prod_{w \in W} S_w$ be the Lie algebra homomorphism given by :

$$f((l_i)) = f(\sum_{i=1}^{p} l'_i) = \prod_{w \in W} l'_w,$$ 

where $\{S'_w\}$ is the collection of inverse limits of inverse systems $\{S'_i\}$. $f$ is well defined. Moreover, if $\{l'_i\} = (\sum_{i=1}^{p} l'_i) \in \text{Ker} f$, then, $f((l'_i)) = (0)$, i.e.
\[
\prod_{w \in A} x_w^{t_d} = 0.
\]
Hence each \( S^{t_d}_w \) = 0 for all \( t \) and for all \( i \). Thus \( (I_t) = 0 \) and \( f \) is injective.

In addition, let \( s = \prod_{w \in A} x_w^{t_d} \) be an element of \( \prod_{w \in A} S^{t_d}_w \), and let \( I = (I_t) \) where \( I_t \) is the finite sum of the the projection of the different \( S^{t_d}_w \). Then \( f(I) = s \) and \( f \) is surjective. Therefore, \( f \) is a Lie algebra isomorphism and \( L \cong \prod_{w \in A} S^{t_d}_w \).

**FINITE CODIMENSIONAL IDEALS OF PROSEMISIMPLE LIE ALGEBRAS**

**Lemma 1.** Let \( L = \prod_{i \in \mathbb{N}} L_i \), \( i \in \mathbb{N} \), where the \( L_i \) are finite dimensional simple Lie algebras over \( K \). If \( A \) is an ideal of \( L \) of finite codimension, then \( A \) contains all but a finite number of the \( L_i \).

**Proof.** The proof will be done by contradiction. Suppose we have an infinite sequence \( L_1, L_2, L_3, \ldots \) of the \( L_i \) not in \( A \). For an arbitrary \( L_i \) of the \( L_i \) in our sequence, \( A \cap L_i \) is an ideal of \( L_i \). Thus \( A \cap L_i \neq \{0\} \) or \( A \cap L_i = L_i \), which is not the case since \( L_i \) is not in \( A \).

Thus, \( A \cap L_i = \{0\} \). Let \( d = \dim L/A \) and let \( V = L_1 \oplus L_2 \oplus \ldots \oplus L_t \) be such that \( \dim V > d \). Let \( \pi' : V \to L/A \) be the restriction of the canonical projection \( L \to L/A \) to \( V \).

Ker \( \pi' \) is an ideal of \( V \); hence, Ker \( \pi' \) is a direct sum of some \( L_i \) in \( V \), say, Ker \( \pi' = L_1 \oplus L_2 \oplus \ldots \oplus L_m \), \( 1 \leq m \leq t \). Then, \( \pi'(L_1) = \pi'(L_2) = \ldots = \pi'(L_m) = 0 \).

Thus, \( L_1, L_2, \ldots, L_m \) would all be included in \( A \), which is not the case. Hence, Ker \( \pi' \) = \( \{0\} \) and consequently we get a contradiction, since dim \( V > \dim L/A \). Thus, \( A \) contains all but a finite number of the \( L_i \).

Let \( x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) be the standard basis of \( S(2,K) \). Then, \([x,y]=h\), \([h,x]=2x\) and \([h,y]=-2y\).

**Lemma 2.** Let \( L = \prod_{i \in \mathbb{N}} L_i \), \( i \in \mathbb{N} \), where the \( L_i \) are finite dimensional simple Lie algebras over \( K \). Suppose that the dimensions of all the \( L_i \) are bounded and let \( A \) be an ideal of \( L \) that contains the elements \( \overline{x_1} = (k_1^0 x_1, k_1^1 x_1, \ldots), \overline{x_2} = (k_2^0 x_2, k_2^1 x_2, \ldots), \ldots, \overline{x_r} = (k_r^0 x_1, k_r^1 x_1, \ldots) \) where the \( k_i^j \) are non-zero constants, then \( A = L \).

**Proof.** According to Theorem 1, each \( L_i \) is of the form \( L_{i_1}^1 + L_{i_2}^2 + \ldots + L_{i_t}^t \) where each \( L_{i_j}^j \) is a sub-Lie algebra of \( L_i \) isomorphic to \( S(2,K) \). Since the dimensions of all the \( L_i \) are
bounded, we can assume without loss of generality that all the $t_i$ are equal to a certain integer $t$.

For an arbitrary element $l$ in $L$,

$$l = (t_0^n, l_0^n, t_1^n, l_1^n, \ldots) + (t_0^{n'}, l_0^{n'}, t_1^{n'}, l_1^{n'}, \ldots) + \ldots + (t_0^{n''}, l_0^{n''}, t_1^{n''}, l_1^{n''}, \ldots).$$

Thus,

$$l = (\alpha_0^n x_1 + \beta_0^n y_1 + \gamma_0^n h_1, \alpha_1^n x_1 + \beta_1^n y_1 + \gamma_1^n h_1, \ldots) + (\alpha_0^{n'} x_2 + \beta_0^{n'} y_2 + \gamma_0^{n'} h_2, \alpha_1^{n'} x_2 + \beta_1^{n'} y_2 + \gamma_1^{n'} h_2, \ldots) + \ldots + (\alpha_0^{n''} x_t + \beta_0^{n''} y_t + \gamma_0^{n''} h_t, \alpha_1^{n''} x_t + \beta_1^{n''} y_t + \gamma_1^{n''} h_t, \ldots),$$

where the $\alpha_i^n, \beta_i^n$ and $\gamma_i^n$ are constants.

For arbitrary integer $n$, $1 \leq n \leq t$, let

$$[\alpha_0^n x_n, \alpha_1^n x_n] = \left(\frac{\alpha_0^n h_n; \alpha_1^n h_n}{2k_0^n}; \ldots \right) \in L.$$

Since $A$ is an ideal of $L$, $[\bar{y}_a, \bar{y}_b] = \left(\frac{\gamma_0^n h_n; \gamma_1^n h_n}{2k_1^n}; \ldots \right) \in A$. Let

$$\bar{y}_a = \left(\frac{\gamma_0^n h_n; \gamma_1^n h_n}{2k_1^n}; \ldots \right) \in L.$$ Then, $[\bar{y}_a, \bar{y}_b] = \left(\frac{\gamma_0^n h_n; \gamma_1^n h_n}{2k_1^n}; \ldots \right) \in A$. Let

$$\bar{y}_a = \left(\frac{\beta_0^n h_n; \beta_1^n h_n}{2k_1^n}; \ldots \right) \in L.$$

Then $[\bar{y}_a, \bar{y}_b] = \left(\frac{\beta_0^n h_n; \beta_1^n h_n}{2k_1^n}; \ldots \right) \in A$.

Hence $A$ contains

$$[\alpha_0^n x_n; \alpha_1^n x_n] = (\gamma_0^n h_n; \gamma_1^n h_n) = ([\alpha_0^n x_n; \beta_0^n y_n; \gamma_0^n h_n] + \alpha_1^n x_n) + \beta_1^n y_n + \gamma_1^n h_n, \ldots) =\left(\frac{\alpha_0^n h_n; \alpha_1^n h_n}{2k_1^n}; \ldots \right) \in A.$$

This being true for all $1 \leq n \leq t$, we conclude that

$$l = (l_0^n, l_1^n, \ldots) + (l_0^{n'}, l_1^{n'}, \ldots) + \ldots + (l_0^{n''}, l_1^{n''}, \ldots) \in A,$$

and $A = L$.

**Theorem 2.** Let $L = \prod L_i$, $i \in N$, where the $L_i$ are finite dimensional simple Lie algebras over $K$. Suppose that the dimensions of all the $L_i$ are bounded and let $A$ be an ideal of $L$ of finite codimension then $L/A$ is semi simple.

**Proof.** As mentioned previously, each $L_i$ is of the form $L_1^i + L_2^i + \ldots L_r^i$ where each $L_j^i$ is a sub-Lie algebra of $L_i$ isomorphic to $\text{S}1(2, K)$. We can assume without loss of generality that all the $t_i$ are equal to a certain integer $t$.

We will first prove that if $J$ is a confinite dimensional ideal of $L$ and $J$ contains every $L_i$, then $J = L$.
This is done as follows: Let \( n = \dim L/ J \). Since \( J \) contains every \( L_i \) and every \( L_i \) contains \( \bigoplus L_i \). For every integer \( j, 1 \leq j \leq t \), let \( S_j \) be the subspace of \( L \) spanned by the vectors \( \{ (x_j; x_j; x_j; \ldots); (2x_j; 3x_j; \ldots); (2^n x_j; 3^n x_j; \ldots) \} \).

These vectors are linearly independent:

If \( \alpha_0 (x_j; x_j; \ldots) + \alpha_1 (2x_j; 3x_j; \ldots) + \ldots + \alpha_n (2^n x_j; 3^n x_j; \ldots) = 0 \) Then,

\[
\begin{align*}
\alpha_0 x_j + 2\alpha_1 x_j + \ldots + 2^n \alpha_n x_j &= 0; \\
\alpha_0 x_j + 3\alpha_1 x_j + \ldots + 3^n \alpha_n x_j &= 0; \\
\alpha_0 x_j + (n+2) \alpha_1 x_j + \ldots + (n+2)^n \alpha_n x_j &= 0.
\end{align*}
\]

i.e.

\[
\begin{pmatrix}
1 & 2 & 2^n \\
1 & 3 & 3^n \\
1 & (n+2) & (n+2)^n
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_n
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

But the first matrix is a Vandermonde matrix so it is invertible and hence \( \alpha_0 = \alpha_1 = \ldots = \alpha_n = 0 \). Thus, the vectors are linearly independent and they form a basis of \( S_j \). Thus \( \dim S_j = n+1 \). But \( \dim L/ J = n \), which implies that \( J \cap S_j \) is non-zero.

Let \( a \) be a non-zero element of the intersection of \( J \) and \( S_j \). Then,

\[
a = (\gamma_0 (x_j; x_j; \ldots) + \gamma_1 (2x_j; 3x_j; \ldots) + \ldots + \gamma_n (2^n x_j; 3^n x_j; \ldots)) = (\gamma_0 x_j + \gamma_1 x_j + \ldots + \gamma_n x_j) \gamma_0 + \gamma_1 \gamma_2 + \ldots + \gamma_n \gamma_0 x_j + \gamma_0 \gamma_2 x_j + \ldots + \gamma_n \gamma_0 x_j.
\]

Thus, \( a = (\beta_0 x_j; \beta_1 x_j; \ldots; \beta_n x_j) = (\beta_m x_j) \in \mathbb{N} \), where

\[
\begin{align*}
\beta_0 &= \gamma_0 + 2\gamma_1 + \ldots + 2^n \gamma_n; \\
\beta_1 &= \gamma_0 + 3\gamma_1 + \ldots + 3^n \gamma_n; \\
\beta_n &= \gamma_0 + (n+1)\gamma_1 + \ldots + (n+1)^n \gamma_n.
\end{align*}
\]

Now, if there exist \( (n+1) \) or more \( \beta_m \) which are zeros, then, assume without loss of generality that \( \beta_0 = \beta_1 = \ldots = \beta_n = 0 \). Then,

\[
\gamma_0 + 2\gamma_1 + \ldots + 2^n \gamma_n = 0; \\
\gamma_0 + (n+2) \gamma_1 + \ldots + (n+2)^n \gamma_n = 0.
\]

i.e.

\[
\begin{pmatrix}
1 & 2 & 2^n \\
1 & 3 & 3^n \\
1 & (n+2) & (n+2)^n
\end{pmatrix}
\begin{pmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_n
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

But the first matrix is a Vandermonde matrix which is invertible, thus \( \gamma_0 = \gamma_1 = \ldots = \gamma_n = 0 \) and \( a = 0 \), contradicting the assumption that \( a \neq 0 \). Therefore, \( \beta_m = 0 \) for at most \( n \) \( \beta_m \). Let \( k \) be the number of entries where \( \beta_m = 0 \), \( 1 \leq k \leq n \), we can assume without loss of generality that the only entries of \( a \) for which \( \beta_m = 0 \) are the first \( k \) entries. Since \( J \)
contains $\bigoplus L_i$, $J$ contains the element $a=\langle x_1,x_2,...,x_t,0,0,...\rangle$, where only $k$ entries are $x_j$ and the rest are zeros. Thus $J$ contains the element $a+a'=\langle x_1,x_2,...,x_t,\beta_{k+1},x_{t+1},...\rangle$. Since this is true for all $1\leq j \leq t$, it follows by Lemma 2 that $J = L$.

Now, back to our problem. $A$ is a cofinite dimensional ideal of $L$. By Lemma 1, $A$ contains all but a finite number of the $L_i$, say, all but $p$-ones of $L_i$. Let $V$ be the direct sum of these $p$-ones. Then $A \bigoplus V$ contains all the $L_i$, and it is of finite co-dimension. Thus, by what we proved at the beginning, $A \bigoplus V = L$. $L/A \cong V$. Hence, $L/A$ is a product of a finite number of copies of $L_i$.

**Corollary 1.** Let $L = \lim L_i$, $i \in \mathbb{N}$, where each $L_i$ is a finite dimensional semisimple Lie algebra. Suppose that the dimensions of all simple factor ideals of all $L_i$ are bounded. Then every finite dimensional homomorphic image of $L$ is semisimple.

**Proof.** The proof follows directly by combining Proposition 1 and Theorem 2.

**Corollary 2.** Let $L = \prod L_i$, $i \in \mathbb{N}$, where the $L_i$ are finite dimensional simple Lie algebras over $K$. Suppose that the dimensions of all the $L_i$ are bounded, and let $W$ be a finite dimensional Lie algebra over $K$. Suppose $f: L \rightarrow W$ is a Lie algebra homomorphism such that $f(L_i) = 0$ for every $L_i \subset L$. Then $f(\prod L_i) = 0$.

**Proof.** $Ker f$ is an ideal of $L$. $L / Ker f \cong \text{Image } f \subset W$, but $W$ is of finite dimension. Thus $Ker f$ is a finite codimensional ideal of $L$. Moreover, $Ker f$ contains every $L_i \subset L$. Hence, by first part of Theorem 2, $Ker f = \prod L_i$, and $f=0$.

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**References**


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