UTILIZING DEA AND PREFERENCE RELATION FOR COMPARING EFFICIENT DECISION MAKING UNITS: AN APPLICATION FOR RANKING BANK BRANCHES

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ABSTRACT

The aim of this paper is to present an integrated method using data envelopment analysis (DEA) and preference relation for full ranking efficient decision making units (DMUs). One of the main imperfections in using DEA to construct the preference relation is the presence of alternative optimal solutions in the related DEA models. In this situation, ranking DMUs may be varied by changing optimal solution. This fact is shown by an example. In this paper a model based on DEA to acquire the components of the preference relation is derived and then a new method to modify the proposed model in order to obtain unique optimal weights is suggested. The performance of the new method and other methods in the literature are compared by a numerical example. In addition, a numerical example is provided to illustrate the geometrical interpretation of the proposed method. A case study about bank branches in Iran is outlined to assess the validity of the proposed approach.

Keywords: data envelopment analysis, preference relation, alternative optimal solution, ranking

INTRODUCTION

In a recent work, Wu (2009) suggested an integrated method by data envelopment analysis (DEA) and preference relation for full ranking of decision making units (DMUs). Preference relation is a strong tool to select a decision from a set of alternative decisions. In this method, a decision maker (DM) constructs a preference relation based on pairwise comparison between alternatives, which is shown by a matrix named preference matrix. This means that the *i-j*th component of this matrix is the preference of the *i*th decision over the *j*th decision. Then, a priority vector is obtained from the preference relation (matrix) for ranking decisions. The *i*th component of the priority vector is the value of *i*th decision in relation to the rest of decisions. Two preference relations are usually used: multiplicative preference relation that was presented by Saaty (1980) and fuzzy preference relation that was introduced by Orlovsky (1978). The related definitions to the preference relation are discussed in the second section.

However, classical techniques used to construct a preference relation are based on subjective evaluation, requiring much involvement of expert knowledge and time (Wu, 2009).

DEA is a potent tool to provide objective information to construct preference matrices. This method utilizes a mathematical programming technique to evaluate the relative efficiency of homogeneous decision making units (DMUs) on the basis of multiple inputs and outputs as suggested by Charnes *et al.* (1978) (CCR model). DEA is a widely used method in management science, operational research, engineering systems, decision analysis and so on. A thorough review on DEA was done by Cook and Seiford (2009).

One of the defects of the suggested DEA models to construct the preference relation is the presence of alternative optimal solutions. Alternative optimal solutions may lead to different preference matrices that produce different priority vectors. Diverse priority vectors may yield different ranks for each DMU. In this paper, a new model based on DEA is suggested to construct the preference matrix. The benefit of the method is to obtain a unique optimal solution from the related DEA model. By using the proposed approach, not only realistic and full ranking of the efficient DMUs is obtained (defection of the conventional DEA models is removed), but also the difficulty of the alternative optimal solutions is deleted. To show the ability of the proposed approach, a case study about Iranian banks is considered.

The rest of the paper is organized as follows. In the second section, the preference relation method is described. A new DEA method is proposed to construct the preference relation in the third section. In the fourth section, a numerical example is considered for comparing the proposed method in this paper with the method suggested by Wu (2009). Furthermore, a numerical example is provided to describe a geometrical interpretation of the proposed method. In this section, this method is also applied to rank bank branches in Iran and the fifth section concludes the paper.

DECISION MAKING BY PREFERENCE RELATION

Assume there are n decisions. The Preference relation method is an important method to select a decision from among a set of alternative decisions. The preference degree of decision i over decision j is denoted by e_{ij} $(i, j = 1, \dots, n)$. The preference degrees can be obtained by subjective information. Then, the preference matrix $E = (e_{ij})_{n*n}$ is constructed and a priority vector is obtained based on this matrix. The priority vector is used to adopt one of the decisions as the best decision.

In this situation, the question to be addressed is how much the obtained priority vector coïncides with the real priority of decisions. To answer this question, the concept of consistent preference matrix was introduced. Let, $W = (w_i)_{n*1}$ be the real priority vector of decisions. Therefore, w_i / w_j $(i, j = 1, \dots, n)$ is the preference degree of the *i*th decision over the *j*th decision.

Definition 1. Let $E = (e_{ij})_{n*n}$ be a preference matrix, then $E = (e_{ij})_{n*n}$ is a consistent preference matrix if $e_{ij} = w_i / w_j$ $(i, j = 1, \dots, n)$.

Theorem 2. Let $W^* = (w_i^*)_{n*1}$ be the obtained priority vector of the consistent preference matrix $E = (e_{ij})_{n*n}$, then $w_i^* = w_i$ $(i = 1, \dots, n)$.

This theorem states that when preference relation is consistent, the obtained priority of decisions coïncides with the real priority of them. Here, some concepts are presented to construct a consistent preference matrix.

Generation consistent preference matrix

Definition 3. Let $E = (e_{ij})_{n*n}$ be a preference matrix, then $E = (e_{ij})_{n*n}$ is called a fuzzy preference matrix if $e_{ij} \in [0, 1]$ $(i, j = 1, \dots, n)$ and $e_{ij} + e_{ji} = 1$ $(i, j = 1, \dots, n)$.

Definition 4. Let $E = (e_{ij})_{n*n}$ be a preference matrix, then $E = (e_{ij})_{n*n}$ is called multiplicative preference matrix, if $e_{ij} \in R^+$ $(i, j = 1, \dots, n)$ and $e_{ij} = 1/\, e_{ji}$ $(i, j = 1, \dots, n)$.

Theorem 5. Let $E = (e_{ij})_{n*n}$ be a preference matrix, then $R = (r_{ij})_{n*n}$ with $r_{ij} = (e_{ii} + e_{ij})/(e_{ii} + e_{ij} + e_{jj} + e_{ji})$ $(i, j = 1, \dots, n)$ is a fuzzy preference matrix.

Theorem 6. Let $E = (e_{ij})_{n*n}$ be a preference matrix, then $S = (s_{ij})_{n*n}$ with $s_{ij} = (e_{ii} + e_{ij})/(e_{jj} + e_{ji})$ $(i, j = 1, \dots, n)$ is a multiplicative preference matrix.

Theorem 7. Let $R = (r_{ij})_{n*n}$ be a fuzzy preference matrix, then $S = (s_{ij})_{n*n}$ with $s_{ij} = r_{ij}\,/r_{ji}$ $(i, j = 1, \dots, n)$ is a multiplicative preference matrix.

Theorem 8. Let $R = (r_{ij})_{n*n}$ is a fuzzy preference matrix, then $B = (b_{ij})_{n*n}$ with $b_{ij} = (r_i - r_j)/(2 * (n - 1)) + 0.5$ $(i, j = 1, \dots, n)$, which $r_i = \sum_{j=1}^{n} r_{ij}$ $(i = 1, \dots, n)$, is a consistent fuzzy preference matrix.

Based on these discussions, a preference matrix can be converted to a consistent fuzzy preference matrix or consistent multiplicative preference matrix.

Obtaining priority vector

There are several methods to obtain the priority vector of a preference matrix. Row wise summation method, eigenvector method, least square method and logarithmic least square method are more applicable methods to derive the priority vector. For the preference matrix $B = (b_{ij})_{n*n}$, $w_i = \sum_{j=1}^{n} b_{ij} \,/ \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}$ $(i = 1, \dots, n)$ is considered as the priority of the *i*th decision in the row wise summation method. In the eigenvector method, the maximum eigenvalue of the preference matrix $B = (b_{ij})_{n*n}$ is determined. Then, corresponding to this eigenvalue an eigenvector of B is obtained, which is considered as the priority vector. A full detailed discussion about preference relation method can be found in Saaty (1980), Nurmi (1981) and Yager and Kacprzyk (1997).

METHODOLOGY

Data envelopment analysis

In this section, the consistent fuzzy preference relation is used to rank the efficient DMUs. Assume that there are *n* DMUs to be evaluated where each DMU with *m* inputs and *s* outputs. $x_{ij} > 0$ $(i = 1, \dots, m)$ and $y_{rj} > 0$ $(r = 1, \dots, s)$ are the value of the inputs and outputs of $DMU_j (j = 1, \dots, n)$, respectively. The absolute efficiency of DMU_j is defined as

$\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij}$ $(j = 1, \dots, n)$ where u_r, v_i are the assigned weights to the *r*th output and the *i*th input, respectively. In order to determine the performance of DMU_p in relation to the other DMUs, Charnes *et al*. (1978) developed the following well-known CCR model as:

$$\begin{aligned} \theta_{pp}^* = \text{Max} \quad & \sum_{r=1}^{s} u_r\, y_{rp} / \sum_{i=1}^{m} v_i\, x_{ip}, \\ \text{s.t} \quad & \sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij} \le 1, \;\; j = 1, \dots, n, \\ & u_r \ge \varepsilon, r = 1, \dots, s, \\ & v_i \ge \varepsilon, \;\; i = 1, \dots, m, \qquad\qquad (1) \end{aligned}$$

This model is equivalent to the following linear programming problem, which is known as the CCR oriented model, as:

$$\begin{aligned} \theta_{pp}^* = \text{Max} \quad & \sum_{r=1}^{s} u_r\, y_{rp}, \\ \text{s.t} \quad & \sum_{i=1}^{m} v_i\, x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} \le 0, \;\; j = 1, \dots, n, \\ & u_r \ge \varepsilon, r = 1, \dots, s, \\ & v_i \ge \varepsilon, \;\; i = 1, \dots, m, \qquad\qquad (2), \end{aligned}$$

where subscript p represents the evaluating DMU and ε is a non-Archimedean number. The optimal value of this problem is considered as the relative efficiency of DMU_p. DMU_p is called efficient if $\theta_{pp}^* = 1$. Let, $SE = \{p | \theta_{pp}^* = 1, p \in \{1, \dots, n\}\}$ be the set of efficient DMUs.

Utilizing DEA to construct preference matrix

To obtain the preference degrees of all DMUs over the DMU_p, the DMU_p is considered in the best condition of its efficiency and then $(v_1, \dots, v_m, u_1, \dots, u_s)$ must be determined such that the efficiency scores of the rest DMUs can be increased as greater as possible. In this framework, from the problem (1), we have $\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij} \le 1$ $(j \in SE)$. The higher the value of $\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij}$, the better is the performance of DMU_j $(j \in SE)$. However, $\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij} \le 1$ $(j \in SE)$ is equivalent to $(\sum_{r=1}^{s} u_r\, y_{rj} + s_j) / \sum_{i=1}^{m} v_i\, x_{ij} = 1$, where $s_j \ge 0$ is a slack variable for constraint $\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij} \le 1$ $(j \in SE)$. To have the value of $\sum_{r=1}^{s} u_r\, y_{rj} / \sum_{i=1}^{m} v_i\, x_{ij}$ $(j \in SE)$ as greater as is possible, one must minimize slack variable s_j $(j \in SE)$. Therefore, in order to determine the preference degrees of all efficient DMUs simultaneously in relation to the DMU_p, a multiple objective programming problem is presented as follows:

$$\begin{aligned} \text{Min} \quad & s_j, j \in SE, j \ne p, \\ \text{s.t} \quad & (\sum_{r=1}^{s} u_r\, y_{rj} + s_j) / \sum_{i=1}^{m} v_i\, x_{ij} = 1, j \in SE, j \ne p, \\ & \sum_{r=1}^{s} u_r\, y_{rp} / \sum_{i=1}^{m} v_i\, x_{ip} = \theta_{pp}^*, \\ & u_r \ge \varepsilon, r = 1, \dots, s, \\ & v_i \ge \varepsilon, \;\; i = 1, \dots, m, \\ & s_j \ge 0, j \in SE, j \ne p, \qquad\qquad (3) \end{aligned}$$

This problem can be written as:

$$\text{Min} \quad s_j, j \in SE, j \ne p,$$

$$\begin{aligned} \text{s.t} \quad & \textstyle\sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} + s_j = 0, \;\; j \in SE, j \neq p, \\ & \textstyle\sum_{r=1}^{s} u_r\, y_{rp} - \theta_{pp}^{*} \sum_{i=1}^{m} v_i\, x_{ip} = 0, \\ & u_r \geq \varepsilon, r = 1, \dots, s, \\ & v_i \geq \varepsilon, \; i = 1, \dots, m, \\ & s_j \geq 0, j \in SE, j \neq p, \qquad (4) \end{aligned}$$

By applying a widely used method (minsum method) to solve multiple objective programming problems, the above multiple objective linear programming problem is transformed to a linear programming problem as follows:

$$\begin{aligned} \text{Min} \quad & \textstyle\sum_{j \in SE, j \neq p} s_j \\ \text{s.t} \quad & \textstyle\sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} + s_j = 0, \;\; j \in SE, j \neq p, \\ & \textstyle\sum_{r=1}^{s} u_r\, y_{rp} - \theta_{pp}^{*} \sum_{i=1}^{m} v_i\, x_{ip} = 0, \\ & u_r \geq \varepsilon, r = 1, \dots, s, \\ & v_i \geq \varepsilon, \; i = 1, \dots, m, \\ & s_j \geq 0, j \in SE, j \neq p, \qquad (5) \end{aligned}$$

Suppose, $\left(v_1^{*p}, \dots, v_m^{*p}, u_1^{*p}, \dots, u_s^{*p}\right)$ is the optimal weights of the problem (5). Then $\sum_{i=1}^{m} u_r^{*p}\, y_{rj} / \sum_{r=1}^{s} v_i^{*p}\, x_{ij}$ $(j \in SE)$ is considered as the preference degree of DMU_j $(j \in SE)$ over DMU_p. These scores are placed in the *p*th column of a preference matrix. The problem (5) is solved for each efficient DMU. Then a preference matrix is constructed. This preference matrix, by theorem 5, is then transformed to a fuzzy preference matrix, and ,by theorem 8, is converted to a consistent preference matrix. Then the priority vector is derived from this consistent preference matrix, by one of the mentioned methods seen previously. However, there is no guarantee that the optimal solution of problem (5) is unique. Alternative optimal solutions may lead to different priority vectors. Therefore, a specific DMU may have a different rank for each case.

To obtain a unique optimal solution of problem (5), the following method is suggested.

Unique priority vector

Consider a linear programming problem and its dual as:

$$\begin{aligned} & Min \;\; cx & & & & Max \;\; wb \\ & s.t \;\;\; Ax = b, & & (6), \qquad & & s.t \;\;\; wA \leq c, \;\; (7), \\ & \qquad\;\; x \geq 0, & & & & \qquad w \;\; free, \end{aligned}$$

where A is an $m \times n$ matrix and rank (A) = *m*. All definitions and theorems are taken from Murty (2002).

Definition 9. A basic feasible solution of problem (6) is degenerate if at least one of its basic variables is zero. A basic feasible solution is nondegenerate if it is not degenerate.

Definition 10. The linear programming problem (6) is totally nondegenerate if each basic feasible solution is non-degenerate.

Theorem 11. The linear programming problem (6) is totally nondegenerate iff each feasible solution has at least *m* non-zero components.

Theorem 12. If a linear programming problem has alternative optimal solutions, its dual has a degenerate optimal solution.

From the mathematical standpoint, alternative optimal solutions of the primal lead to the degeneracy in the optimal solution of the dual problem. Therefore, if one avoids the degeneracy in optimal solutions of the dual, one can remove the alternative optimal solutions of the primal model. To do so, one proceeds as follows.

Theorem 13. For each $b \in R^m$ there is a positive real number ε_1, such that for all $\bar{\varepsilon}$ and $0 < \bar{\varepsilon} < \varepsilon_1$ the problem

$$Min\ cx\ \ s.t\ \ Ax = b(\varepsilon) = (b_1 + \bar{\varepsilon}^1, \ldots, b_m + \bar{\varepsilon}^m), x \geq 0, \qquad (8),$$

is totally nondegenerate.

By contra positive of theorem (12), if a linear programming problem does not have a degenerate optimal solution, then its dual does not have alternative optimal solutions (has a unique optimal solution).

Result 14. The problem (8) is a nondegenerate linear programming problem, so it does not have a degenerate optimal solution and therefore, the dual of the problem (8) does not have alternative optimal solutions.

Now, consider the dual of problem (5) as:

$$\begin{aligned}
\text{Max} \quad & \varepsilon\left(\textstyle\sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r\right), \\
\text{s.t} \quad & \textstyle\sum_{j \in SE, j \neq p} \gamma_j x_{ij} + \theta_{pp}^* \gamma_p x_{ip} - \alpha_i = 0, \ \ i = 1, \ldots, m, \\
& \textstyle\sum_{j \in SE, j \neq p} \gamma_j y_{rj} + \gamma_p y_{rp} + \beta_r = 0, \ \ r = 1, \ldots, s, \\
& \gamma_j \leq 1, j \in SE, j \neq p, \\
& \alpha_i \geq 0, \ \ i = 1, \ldots, m, \\
& \beta_r \geq 0, r = 1, \ldots, s, \qquad (9)
\end{aligned}$$

Using theorem (13), there is an ε_1 , such that for all $0 < \bar{\varepsilon} < \varepsilon_1$, the following problem does not have degenerate optimal solution.

$$\begin{aligned}
\text{Max} \quad & \varepsilon\left(\textstyle\sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r\right), \\
\text{s.t} \quad & \textstyle\sum_{j \in SE, j \neq p} \gamma_j x_{ij} + \theta_{pp}^* \gamma_p x_{ip} - \alpha_i = \bar{\varepsilon}^i, \ \ i = 1, \ldots, m, \\
& \textstyle\sum_{j \in SE, j \neq p} \gamma_j y_{rj} + \gamma_p y_{rp} + \beta_r = \bar{\varepsilon}^{m+r}, \ \ r = 1, \ldots, s, \\
& \gamma_j \leq 1 + \bar{\varepsilon}^{m+s+j}, j \in SE, j \neq p, \\
& \alpha_i \geq 0, \ \ i = 1, \ldots, m, \\
& \beta_r \geq 0, r = 1, \ldots, s, \qquad (10)
\end{aligned}$$

So, the dual of the problem (10) does not have alternative optimal solutions. The dual is as:

$$\begin{aligned}
\text{Min} \quad & \textstyle\sum_{j\in SE, j\neq p} s_j + \sum_{i=1}^{m} \bar{\varepsilon}^{i} v_i + \sum_{r=1}^{s} \bar{\varepsilon}^{m+r} u_r + \sum_{j\in SE, j\neq p} \bar{\varepsilon}^{m+s+j} s_j \\
\text{s.t} \quad & \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + s_j = 0, \ \ j \in SE, j \neq p, \\
& \textstyle\sum_{r=1}^{s} u_r y_{rp} - \theta_{pp}^{*} \sum_{i=1}^{m} v_i x_{ip} = 0, \\
& u_r \geq \varepsilon, r = 1, \dots, s, \\
& v_i \geq \varepsilon, \ i = 1, \dots, m, \\
& s_j \geq 0, j \in SE, j \neq p, \qquad (11)
\end{aligned}$$

Therefore, in order to obtain a unique set of weights, solving model (11) is suggested.

In the next section, several numerical examples are presented to state the abilities of the proposed method.

NUMERICAL EXAMPLES

In this section, three numerical examples are provided. The first example compares the proposed method in this paper with the method of Wu (2009). This example is also used to show the caused problem for ranking DMUs when DEA model has alternative optimal solutions. The second example discusses the proposed method in this paper for ranking the efficient DMUs. In the next example, an empirical study about 22 bank branches in Iran is carried out and the performances of the efficient branches are evaluated by the proposed method.

Comparison example

In this example, four DMUs with two inputs and two outputs, which are reported in Table 1, are considered to compare the proposed method in this paper with the method of Wu (2009). In this example, all DMUs are efficient. Wu (2009) used the following DEA model for constructing a preference matrix as:

$$\begin{aligned}
\text{Max} \quad & \textstyle\sum_{r=1}^{s} u_r \sum_{j\in SE, j\neq p} y_{rj}, \\
\text{s.t} \quad & \textstyle\sum_{i=1}^{m} v_i \sum_{j\in SE, j\neq p} x_{ij} = 1, \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j \in SE, j \neq p, \\
& \textstyle\sum_{r=1}^{s} u_r y_{rp} - \theta_{pp}^{*} \sum_{i=1}^{m} v_i x_{ip} = 0, \\
& u_r \geq \varepsilon, r = 1, \dots, s, \\
& v_i \geq \varepsilon, \ i = 1, \dots, m, \qquad (12)
\end{aligned}$$

TABLE 1

DMUs' Data

DMU	Input 1	Input 2	Output 1	Output 2	CCR efficiency
DMU_1	520	2.0	52500	32000	1
DMU_2	130	0.5	13250	2000	1
DMU_3	370	1.0	7250	20000	1
DMU_4	280	1.5	32000	10000	1

Model (12) has alternative optimal solutions, for *p=1, 2*. As shown in the second row of Tables 2 and 3, $(v_1, v_2, u_1, u_2) = (0.7865489\mathrm{E}-02, 0.9549729, 0.1128527\mathrm{E}-03, 0.2351097\mathrm{E}-05)$ and $(v_1, v_2, u_1, u_2) = (0.6585797\mathrm{E}-02, 1.287693, 0.0.1\mathrm{E}-05, 0.1858594\mathrm{E}-03)$ are alternative solutions of the problem (12), for *p=1*. According to the fourth and fifth rows of the Tables 2 and 3, model (12) has a unique optimal solution, for *p=3, 4*. By using weights in these two Tables, two preference matrices are obtained, which are shown by E_1 and E_2. These preference matrices are transformed to fuzzy preference matrices. Two different priority vectors are obtained from these fuzzy preference matrices. Priority vector by weights in Table 2 is $W_1 = (0.2600447, 0.2526198, 0.2352368, 0.2520987)$ and so DMUs are ranked as $\mathrm{DMU}_1 \gg \mathrm{DMU}_2 \gg \mathrm{DMU}_4 \gg \mathrm{DMU}_3$. On the other hand, priority vector by weights in Table 3 is $W_2 = (0.2654739, 0.2420563, 0.2442144, 0.2482554)$ and so DMUs are ranked as $\mathrm{DMU}_1 \gg \mathrm{DMU}_4 \gg \mathrm{DMU}_3 \gg \mathrm{DMU}_2$. Although, DMU_1 has the first rank, by both priority vectors, the rest of DMUs has different ranks by these two priority vectors. Therefore there is an ambiguity in ranking DMUs. Thus, a method is needed to acquire a unique set of weights, corresponding to each DMU.

TABLE 2

The First Optimal Weights

DMU	v_1	v_2	u_1	u_2
DMU_1	0.7865489E-02	0.9549729	0.1128527E-03	0.2351097E-05
DMU_2	0.1000000E-05	1.999740	0.7523511E-04	0.1567398E-05
DMU_3	0.1000000E-05	2.249767	0.2200086E-04	0.1045316E-03
DMU_4	0.8823526E-02	0.1000000E-05	0.6648594E-04	0.3430386E-04

TABLE 3

The Second Optimal Weights

DMU	v_1	v_2	u_1	u_2
DMU_1	0.6585797E-02	1.287693	0.1000000E-05	0.1858594E-03
DMU_2	0.5243659E-02	0.6366487	0.7523511E-04	0.1567402E-05
DMU_3	0.1000000E-05	2.249767	0.2200086E-04	0.1045316E-03
DMU_4	0.8823526E-02	0.1000000E-05	0.6648594E-04	0.3430386E-04

$$\mathrm{E}_1 = \begin{bmatrix} 1.000000 & 0.85714 & 0.35821 & 1.000000 \\ 1.000000 & 1.00000 & 0.4449499 & 0.8278097 \\ 0.2238443 & 0.2883854 & 1.000000 & 0.3577965 \\ 1.00000 & 0.8077621 & 0.5183347 & 1.000000 \end{bmatrix},$$

$$\mathrm{E}_2 = \begin{bmatrix} 1.000000 & 1.00000 & 0.35821 & 1.000000 \\ 0.2566458 & 1.00000 & 0.4449499 & 0.8278097 \\ 1.000000 & 0.2238443 & 1.000000 & 0.3577965 \\ 0.5007450 & 1.000000 & 0.5183347 & 1.000000 \end{bmatrix}$$

TABLE 4

Unique Optimal Weights by the Proposed Method

DMU	v_1	v_2	u_1	u_2
DMU_1	0.1624962E-03	0.1000000E-05	0.1000000E-05	0.1000000E-05
DMU_2	0.3345455E-02	0.4061818	0.4800000E-04	0.1000000E-05
DMU_3	0.1000000E-05	0.1024300	0.1000000E-05	0.4777500E-05
DMU_4	0.8823526E-02	0.1000000E-05	0.1938258E-05	0.1000000E-05

By solving this suggested model, the unique set of weights for each DMU is obtained as is reported in Table 4. Therefore, the preference matrix, which is denoted by E, is obtained as follows:

$$\mathrm{E} = \begin{bmatrix} 1.000000 & 1.00000 & 1.00000 & 1.000000 \\ 0.7218935 & 1.00000 & 0.4441523 & 0.8278177 \\ 0.4532257 & 0.2238443 & 1.000000 & 0.3577910 \\ 0.9230682 & 1.000000 & 0.5182789 & 1.000000 \end{bmatrix}$$

The priority vector is as $W = (0.2605395, 0.2484512, 0.2376235, 0.2533858)$. Therefore, unique and full ranking of DMUs is $\mathrm{DMU}_1 \gg \mathrm{DMU}_4 \gg \mathrm{DMU}_2 \gg \mathrm{DMU}_3$.

Illustrative example

Ten DMUs with two inputs and one output are considered in this example. The data are provided in Table 5. Moreover, the CCR efficiencies of the units are presented in the last column of Table 5. Farrell's frontier for these DMUs is also shown in Figure 1. As a result, the units $\mathrm{D}_1, \mathrm{D}_6, \mathrm{D}_7$ and D_8 are CCR efficient. To illustrate the proposed method in this paper, efficient units are considered and ranked.

TABLE 5

DMUs' Data

DMU	Input 1	Input 2	Output	CCR efficiency
D_1	0.5	11	1	1.00000
D_2	2	8	1	0.80769
D_3	8	4	1	0.60000
D_4	4	3	1	0.92308
D_5	4	7	1	0.61765
D_6	1	8	1	1.00000
D_7	9	1	1	1.00000
D_8	3	3	1	1.00000
D_9	6	8	1	0.45652
D_{10}	5	4	1	0.70588

By solving the model (11), the weights corresponding to each efficient DMU are presented in the first row in Table 6. According to the weights, the absolute efficiencies of the efficient DMUs are calculated and presented in Table 6. For example, by solving the model (11) for D_1, the optimal weights are given by $(v_1^*, v_2^*, u^*) = (6E-06, 1E-06, 14E-06)$. By using these weights, the absolute efficiencies of the units $\mathrm{D}_1, \mathrm{D}_6, \mathrm{D}_7$ and D_8 are 1, 1, 0.25455 and 0.66667, respectively. These scores are presented in the second column of Table 6.

TABLE 6

The Results of Solving Problem (11)

DMU	(6E-06,1E-06, 14E-06)	(2.5E-06,1E-06, 10.5E-06)	(1E-06,3E-06, 12E-06)	(1E-06,1E-06, 6E-06)
D_1	1.00000	0.85714	0.35821	0.52174
D_6	1.00000	1.00000	0.48000	0.66667
D_7	0.25455	0.44681	1.00000	0.60000
D_8	0.66667	1.00000	1.00000	1.00000

According to the Table 6, preference matrix is as follows:

$$\mathrm{E} = \begin{bmatrix} 1.00000 & 0.85714 & 0.35821 & 0.52174 \\ 1.00000 & 1.00000 & 0.48000 & 0.66667 \\ 0.25455 & 0.44681 & 1.00000 & 0.60000 \\ 0.66667 & 1.00000 & 1.00000 & 1.00000 \end{bmatrix}$$

Utilizing this preference matrix, fuzzy preference matrix and consistent fuzzy preference matrix are obtained. These matrices are provided in the first and second columns of Table 7. The priority vector of the consistent fuzzy preference matrix is given by $(0.24821, 0.24823, 0.24324, 0.26032)$. By these priorities, one can rank efficient DMUs and have $\mathrm{D}_8 \gg \mathrm{D}_6 \gg \mathrm{D}_1 \gg \mathrm{D}_7$. These results are presented in the last column of Table 7.

TABLE 7

The Results of Applying the Proposed Method

Fuzzy preference matrix				Consistent fuzzy preference matrix				Priority vector	Ranking
0.50000	0.48148	0.51984	0.47727	0.50000	0.49998	0.50994	0.47581	0.24821	3
0.51852	0.50000	0.50567	0.45455	0.50002	0.50000	0.50997	0.47583	0.24823	2
0.48016	0.49433	0.50000	0.44444	0.49006	0.49003	0.50000	0.46587	0.24324	4
0.52273	0.54545	0.55556	0.50000	0.52419	0.52418	0.52418	0.50000	0.26032	1

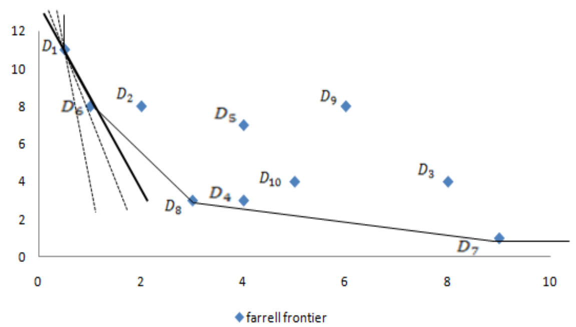

Figure 1. Farrell frontier for data in Table 5.

By solving model (5), a supporting hyperplane of the PPS which passes through the efficient DMU_p is found. If a DMU puts on a facet of the PPS with a dimension less than $m + s$, there are many supporting hyperplanes of the PPS at the DMU. In this situation, the model (5) usually has alternative optimal solutions. As shown in Figure 1, D_1 is an efficient vertex DMU and so many supporting hyperplanes of the PPS pass through this point. Regarding the model (11), for *p=1*, a unique supporting hyperplane of the PPS at D_1 is found, which is shown by solid line in Figure 1.

Empirical example

To state the ability of the proposed method, 22 bank branches in Iran are studied. Seven factors are considered to evaluate these branches, three inputs and four outputs. The inputs are payable interest, staff and non-perform loans. The outputs are loan granted, received interest, fee and total deposits. The data set of branches is presented in Table 8. According to the CCR efficiency of the branches, which are reported in the last column of Table 8, branches B_4, B_5, B_{18}, B_{19}, B_{20} and B_{22} are efficient units. The numerical results of applying the proposed method for ranking these branches are provided in Table 9.

Using the proposed method, consistent fuzzy preference matrix is obtained, which is presented in the second column of Table 9. The priority vector is shown in the third column of Table 9 and the last column of Table 9 has the ranks of the efficient branches. According to this column B_4 has the first rank of among all branches and B_{22} has the last rank among efficient branches.

TABLE 8

The Data Set of Bank Branches in Iran

DMU	Input 1	Input 2	Input 3	Output 1	Output 2	Output 3	Output 4	CCR efficiency
B_1	1639.17	14.21	16675	203538	354.98	138.41	265876	0.615294
B_2	1307.91	12.15	7438	157796.7	728.12	244.29	194615.3	0.802792
B_3	981.03	13.11	10386.79	202161.1	932.39	145.42	177379.7	0.638609
B_4	659.82	15.99	9270.5	651237	1832.74	336.52	168406	1.000000
B_5	2372.7	11.13	19420	570838.3	581.23	225.13	260858.1	1.000000
B_6	1187.25	11.76	5276.28	73978.42	999.96	97.46	172727.7	0.771150
B_7	679.26	11.7	6978	83922	294.46	50.1	158688.5	0.554210
B_8	554.7	11.37	3964	14227	109.04	16.16	74457	0.283633
B_9	1510.91	20.3	2888	93208	568.77	96.85	193930	0.654061
B_{10}	563.41	17.17	9941.5	208500	135.79	14.22	85743	0.364077
B_{11}	372.96	17.47	22019	24035	1534.69	19.82	83168	0.539330
B_{12}	817.74	11.61	8749.71	99227.71	326.75	48.79	119130.1	0.441226
B_{13}	651.69	9.71	20447	75052	284.04	26.47	91971	0.408126
B_{14}	437.36	16.57	12440	182611	391.84	163.96	367828	0.861391
B_{15}	493.44	14.19	3995	93974	476.6	39.86	93984	0.453628
B_{16}	505.24	16.23	9308.33	105776.7	950	358.9	162346.7	0.597814
B_{17}	173.51	16.32	7507	30231	351.65	24.02	77586	0.578643
B_{18}	525.47	9.86	174	71825	1023.1	90.95	95792	1.000000
B_{19}	26.75	5.68	2705	32691	3.84	26.43	158786	1.000000
B_{20}	570.79	17.7	1649	48989	242.37	145.38	157889	1.000000
B_{21}	680.43	17.74	42614.33	68866	1427.05	310.97	126104	0.268390
B_{22}	381.02	19.99	478.66	99312.33	544.35	39.82	73950.65	1.000000

TABLE 9

The Results of Applying the Proposed Method

Efficient DMU	Consistent fuzzy preference matrix						Priority vector	Ranking
B_4	0.500000	0.536914	0.509893	0.500806	0.555428	0.568030	0.176198	1
B_5	0.463086	0.500000	0.472979	0.463891	0.518514	0.531116	0.163858	4
B_{18}	0.490107	0.527022	0.500000	0.490913	0.545535	0.558138	0.172891	3
B_{19}	0.499194	0.536109	0.509087	0.500000	0.554623	0.567225	0.175929	2
B_{20}	0.444572	0.481486	0.454465	0.445378	0.500000	0.512602	0.157669	5
B_{22}	0.431970	0.468884	0.441863	0.432775	0.487398	0.500000	0.153456	6

CONCLUSION

This paper proposed a method for ranking the efficient DMUs by using the DEA and preference relation to obtain full ranking for the efficient DMUs. One of the major imperfections of this method is the presence of alternative optimal solutions in the proposed DEA model. In other words, there is no guarantee that the DEA model has a unique optimal solution. Alternative optimal solutions may lead to different preference matrices. Two different preference matrices may lead to two distinct ranks for each DMU. In this paper, a method to obtain a unique optimal solution from the proposed model based on DEA to construct preference relation was introduced. This technique can be applied for all methods that utilize linear programming problems and need to unique optimal solution. Moreover, a geometrical interpretation of the suggested cross-efficiency method was stated by a numerical example. Furthermore, the proposed method was applied for ranking the efficient branches among 22 bank branches in Iran. In the future, the proposed method in this paper for ranking DMUs with imprecise (fuzzy, stochastic or interval) data will be extended.

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