# EQUIVALENCE BETWEEN TWO DECONVOLUTION TECHNIQUES FOR SISO SYSTEMS

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***ABSTRACT***

*In this paper we demonstrate the equality between two tranfer functions via the application of different approaches. The first approach consists of using the optimal control theory problem; whereas the second consists of using the regularization method. The latter approach leads to a structure identical to Wiener filter. Both approaches are then investigated in the scalar and matricial cases for single-input single-output Systems (SISO).*

**Keywords :** criterion performance, optimal control, transfer function, variational calculus, Wiener filter

## INTRODUCTION

The input - output relation for SISO linear systems is given by the convolution product as follows :

$$y(t)=h(t)*u(t)$$

$$y_m(t)=y(t)+v(t)$$

where * denotes the deconvolution, $h(t)$ is the process impulse response, $u(t)$ is the input signal, $v(t)$ is the noise measurement and $y_m(t)$ is the recorded data.

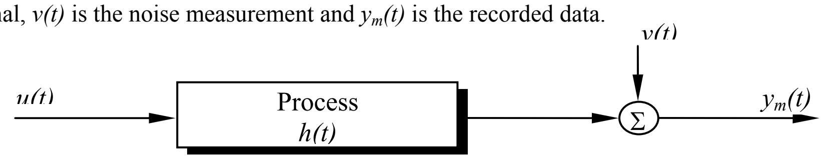


**Figure 1.**

The input estimation termed as deconvolution for linear stationary systems is known to be an ill-posed problem (Tikhonov and Arsenin, 1977). As a matter of fact, singular values in the spectrum of *h(t)* disables a stable direct estimation of *u(t)* since $y_m(t)$ is a noisy signal.

Solutions to this inconvenience lead mostly to stable pseudo-inverse solutions of *h(t)*. Intuitive iterative (van Cittert, 1931), probabilistic (Demoment, 1989) and regularization (Miller, 1970; Tikhonov and Arsenin, 1977) techniques have been proposed to tackle this problem. Among these approaches, the regularized inverse filter is one of the most popular. Recent works have lead to optimal control based estimation algorithms (Thomas, 1980; Sekko, 1996).

In the present paper, equivalence between regularization technique and optimal control is proved. In Section 2, the regularization technique is delineated. The optimal tracking control deconvolution is developed in Section 3. The duality of both methods is proved in Section 4. Two simple examples are given in Section 5 in order to illustrate the equivalence. Finally, concluding remarks are given in Section 6.

**Regularized deconvolution method**

System impulse response spectrum gives rise to singular values precluding direct model inversion. In order to avoid this drawback, Philips (1962), Tikhonov and Arsenin (1977) and Twomey (1963) have considered the input estimate $\hat{u}_\alpha$ to be the minimum of the following criterion :

$$J_1 = \left\| y_m - h * u \right\|^2 + \left\| \sum_i \alpha_i \frac{d^i}{dt^i} u \right\|^2$$

with $\alpha_i \in \Re^+$ the regularization parameters.

It can be clearly seen that $J_1$ reduces to the Least Square Estimation criterion if all regularization parameters are set to zero.

The first term of this criterion represents fidelity of the data. Concerning the second term, two cases are generally considered :

only $\alpha_0 \neq 0$ : large values of $\hat{u}_\alpha$ are penalized ;
only $\alpha_1 \neq 0$ : a smoothing constraint is imposed to $\hat{u}_\alpha$.

In both cases, as $\alpha_i$ decreases to zero, the ill-conditioning amount in the method increases.

In the following, the first case will be considered and the estimate will be the minimum of :

$$\boldsymbol{J}_2 = \left\| \boldsymbol{y_m} - \boldsymbol{h} * \boldsymbol{u} \right\|^2 + \boldsymbol{r} \left\| \boldsymbol{u} \right\|^2$$

$r = \alpha_0^2$

*Theorem 1*

The optimal deconvolution filter minimizing $J_2$ such as :

$$\hat{U}_\alpha(s)=F_r(s)\ Y_m(s)$$

is given by :

$$F_r(s)=\frac{H(-s)}{H(s)H(-s)+r}$$

with $\hat{U}_\alpha(s)$, $Y_m(s)$ the Laplace transform of $\hat{u}_\alpha(t)$ and $y_m(t)$ respectively.

*Proof*

Our aim is to find an optimal estimation of the signal $u(t)$ minimizing the following criterion

$$J(u)=\int_0^{+\infty}\left\{.(y_m(t)-h(t)*u(t))^2+r.u^2(t)\right\}.dt$$

where

$$h(t)*u(t)=\int_{-\infty}^{+\infty}h(t-\tau)u(\tau)d\tau$$

therefore

$$J(u)=\int_0^{+\infty}\left\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-\tau).u(\tau)d\tau)^2+r.u^2(t)\right\}.dt$$

using the fundamental theorem of variational calculus, taking $u=\hat{u}+\Delta u$ we can write :

and finally

$$J(\hat{u}+\Delta u)=\int_0^{+\infty}\left\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-\tau)[\hat{u}(\tau)+\Delta u(\tau)]d\tau)^2+r.[\hat{u}(t)+\Delta u(t)]^2\right\}dt$$

$$=\int_0^{+\infty}\left\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-\tau)\hat{u}(\tau)\,d\tau-\int_{-\infty}^{+\infty}h(t-\tau)\Delta u(\tau)\,d\tau)^2+r.[\hat{u}^2(t)+2.\hat{u}(t)\,\Delta u(t)+\Delta u^2(t)]\right\}.dt$$

$$J(\hat{u}+\Delta u)=\int_{0}^{+\infty}\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-\tau)\hat{u}(\tau).d\tau)^2+r.\hat{u}^2(t)\}.dt+$$

$$\int_{0}^{+\infty}\{(\int h(t-\tau)\Delta u(\tau)d\tau)^2+r.\Delta u^2(t)\}.dt$$

$$-2\int_{0}^{+\infty}\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-\tau)\hat{u}(\tau).d\tau)(\int_{-\infty}^{+\infty}h(t-\tau)\Delta u(\tau).d\tau)-r.\hat{u}(t)\Delta u(t)\}.dt$$

knowing that $J(\hat{u}+\Delta u)$ is under the classical form :

$$J(\hat{u}+\Delta u)=J(\hat{u})+\delta J(\hat{u}+\Delta u)+\theta(\hat{u}+\Delta u)$$

where

$$\delta J(\hat{u}+\Delta u)=-2\int_{0}^{+\infty}\left\{(y_m(t)-\int_{-\infty}^{+\infty}h(t-v)\hat{u}(v).dv)(\int_{-\infty}^{+\infty}h(t-\tau)\Delta u(\tau).d\tau)-r.\hat{u}(t).\Delta u(t)\right\}.dt$$

or

$$\Delta u=\delta*\Delta u=\int_{-\infty}^{+\infty}\delta(t-\tau)\Delta u(\tau)d\tau$$

where $\delta$ is the Dirac function , this will lead us to write :

$$\delta J(\hat{u}+\Delta u)=-2\int_{0}^{+\infty}\{(y_m(t)\int_{-\infty}^{+\infty}h(t-\tau)\Delta u(\tau).d\tau)-\int_{-\infty}^{+\infty}h(t-v)\hat{u}(v).dv\int_{-\infty}^{+\infty}h(t-\tau)\Delta u(\tau).d\tau$$

$$-r.\hat{u}(t)\int_{-\infty}^{+\infty}\delta(t-\tau)\Delta u(\tau)d\tau\}.dt$$

therefore

$$\delta J(\hat{u}+\Delta u)=-2\int_{0}^{+\infty}\left\{\int_{-\infty}^{+\infty}y_m(t)h(t-\tau)\Delta u(\tau)-\int_{-\infty}^{+\infty}(\int_{-\infty}^{+\infty}h(t-v)\hat{u}(v)dv)h(t-\tau)\Delta u(\tau)-\int_{-\infty}^{+\infty}r\hat{u}(t)\delta(t-\tau)\Delta u(\tau)d\tau\right\}dt$$

$$\delta J(\hat{u}+\Delta u)=-2\int_{0}^{+\infty}\int_{-\infty}^{+\infty}[y_m(t)h(t-\tau)-(\int_{-\infty}^{+\infty}h(t-v)\hat{u}(v)dv)h(t-\tau)-r.\hat{u}(t)\delta(t-\tau)]\Delta u(\tau)\,d\tau.dt$$

with Fubbuni theorem, we write :

$$\delta J(\hat{u}+\Delta u)=-2\int_{-\infty}^{+\infty}\int_{0}^{+\infty}[y_m(t).h(t-\tau)-\int_{-\infty}^{+\infty}(h(t-v)\hat{u}(v)dv).h(t-\tau)-r.\hat{u}(t).\delta(t-\tau)]dt.\Delta u(\tau).d\tau$$

using the known following result :

$$\int_{t_1}^{t_2} f(t).\delta h.dt = 0$$

$\forall \ \delta h \Rightarrow f = 0 \quad \forall \ t \in [\ t_1 \quad t_2]$

We have

$$\int_{0}^{+\infty} y_m(t).h(t-\tau)dt - \int_{0}^{+\infty}\int_{-\infty}^{+\infty}(h(t-v)\hat{u}(v).dv).h(t-\tau)dt - \int_{0}^{+\infty} r.\hat{u}(t).\delta(t-\tau).dt = 0$$

$$\int_{0}^{+\infty} y_m(t)h(-(\tau-t)).dt - \int_{0}^{+\infty}(\int_{-\infty}^{+\infty} h(t-v)\hat{u}(v).dv)h(-(\tau-t)).dt - \int_{0}^{+\infty} r.\hat{u}(t)\delta(t-\tau).dt = 0$$

Which can be written in the following form :

$$\int_{0}^{+\infty} y_m(t).h(-(\tau-t)).dt - \int_{0}^{+\infty}(h*\hat{u})(t).h(-(\tau-t)).dt - \int_{0}^{+\infty} r.\hat{u}(t).\delta(t-\tau).dt = 0$$

As :

$$\int_{0}^{+\infty}\int_{-\infty}^{+\infty}(h(t-v)\hat{u}(v).dv).h(-(\tau-t)).dt = \int_{0}^{+\infty}(h*\hat{u})(t).h(-(\tau-t)).dt$$

and putting

$$h^*(\tau - t) = h(-(\tau - t))$$

we reach the following equation

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}(h(t-v)\hat{u}(v)dv).h(-(\tau - t)).dt = (h * \hat{u}) * h^*$$

for physically realizable signals

$$y_m(t) * h^*(\tau) - [(h * \hat{u}) * h^*](\tau) - r.\hat{u}(t) * \delta(-(\tau - t)) = 0$$

In the Fourier space, we have :

$$Y_m.H^* - H.\hat{U}.H^* - r.\hat{U} = 0$$

$$Y_m.H^* - (H.H^* + r).\hat{U} = 0$$

And finally we obtain

$$\frac{\hat{U}}{Y_m} = \frac{H^*}{H.H^* + r}$$

*Remark 1*

The Wiener inverse filter is given by (Wiener, 1949) :

$$F_w(s) = \frac{H(-s)}{H(s)H(-s) + \dfrac{S_v}{S_u}}$$

where $S_v$ and $S_u$ are respectively the Power Density Spectrum of *v(t)* and *u(t)*.

In order to estimate *u(t)*, assumptions should be made on the spectrum of *u(t)*. If one defines $S_u$ such that $S_u=S_v/\alpha_0^2$ then regularization and Wiener methods are equivalent.

**Deconvolution through optimal tracking control**

Optimal tracking control is used in order to apply to the most suitable control a system in order to get the desired output trajectory. Applied to deconvolution, this trajectory can be considered as the measured signal as shown in Figure 2.

Consequently, our aim is to find the optimal control such that :

$$J_3 = \int_0^T \left(e'Qe + u'Ru\right)dt$$

with $Q$ a semi-definite positive matrix of real elements, $R$ a positive scalar and $e'$ the transpose of $e$.

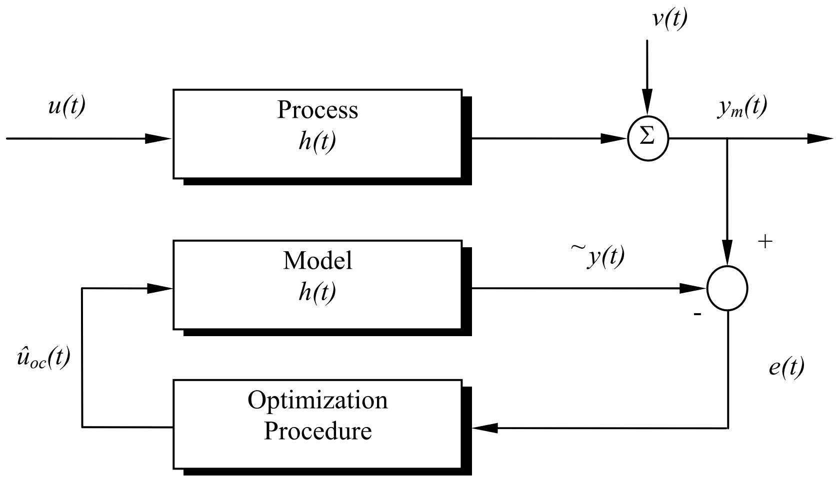


**Figure 2. Optimal tracking control deconvolution scheme.**

*Theorem 2 : Non-stationary deconvolution*

The optimal tracking control minimizing $J_3$ is given by (Thomas, 1980):
with matrices $A \in \Re_{nxn}$, $B \in \Re_{nx1}$ and $C_{1xn} \in \Re$ defined as :

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\hat{u}_{oc}(t) \\ \tilde{y}(t) &= C\tilde{x}(t) \end{aligned} \qquad (1)$$

Let e(t) be the tracking error

$$e(t) = y_m(t) - \tilde{y}(t)$$

Where $y_m(t)$ is the recorded signal, and $y(t)$ the output of the system model whose input is $\hat{u}(t)$. To estimate the quality of the approximation, we define a performance criterion $J_3$.

It can be shown that the optimal control is given by :

$$\hat{u} = -R^{-1}B^t(Kx - v) \tag{2}$$

Where the matrix $K$ is the solution of the Riccati equation :

$$KA + A^tK - KBR^{-1}B^tK + C^tQC = -\dot{K} \tag{3}$$

$v$ is the adjoint vector, solution with n components, of the equation :

$$\dot{v} = -(A^t - KBR^{-1}B^t)v - C^tQ.y_m \tag{4}$$

with the initial condition $v(T)=0$.

*Remark 2*

In the stationary case, one sets $dK(t)/dt$ to zero and solves (3) as an algebraic Riccati Equation.

**Equivalence between regularization and optimal tracking control approach**

The two methods presented in the previous Sections have been developed in two different approaches. As a matter of fact, the inverse regularized filter is obtained in the Laplace domain whereas the optimal control is expressed in the time domain. In order to prove the equivalence of the two methods, the stationary optimal control with $Q=I_{nxn}$ and $R=\alpha_0^2$ will be considered.

Consequently, the equivalence will be established considering the transfer function between the output of the model driven by the estimate, and the measured signal (see figure 3). Equality of the transfer functions will entail the equality of the estimate and $y_m(t)$ for both methods.

*Theorem 3*

The stationary optimal tracking control deconvolution (presented in *Theorem 2* and *Remark 2*, with $Q=I_{nxn}$ and $R=\alpha_0^2$) ; and the regularized inverse technique (in *Theorem 1*) are equivalent.

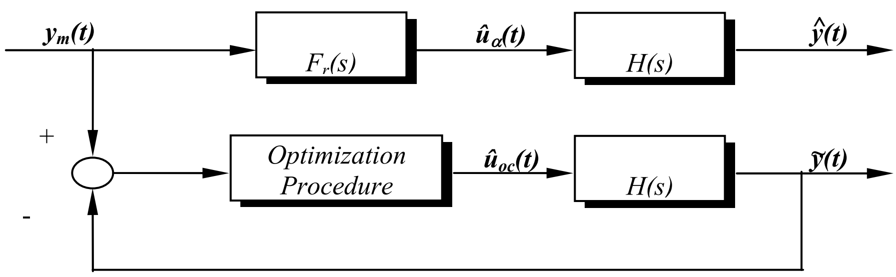


**Figure 3.**

Proof

• The first Step of the proof :

*a-<u>The scalar case with optimal control</u>*

Let's take under consideration the following scalar system :

$$\begin{cases} \dot{x} = ax + bu \\ y = cx \end{cases}$$

Where *a, b*, and *c* are scalars.
The optimal control is given by :

$$\hat{u} = -\frac{b}{r}(Kx + v)$$

then (3) becomes

$$\frac{b^2}{r}k^2 - 2ak - qc^2 = 0 \tag{5}$$

and (4) is written as

$$\dot{v} = -(a - \frac{b^2}{r}k)v + qcy_m \tag{6}$$

with $q$=1, we obtain :

$$k = \frac{a}{b^2}r \pm \frac{r}{b^2}\sqrt{a^2 + \frac{b^2c^2}{r}} \qquad \frac{b^2}{r}k - a = \pm\sqrt{a^2 + \frac{b^2c^2}{r}} \tag{7}$$

in Fourier space, we write :

$$(s - a + \frac{b^2}{r}k)X = -\frac{b^2}{r}V$$

where $X$ and $V$ are the Fourier transforms of $x$ and $v$. Then we obtain with (1)

$$Y = -\frac{1}{r}\frac{cb^2}{(s - a + \frac{b^2}{r}k)}V \tag{8}$$

using (6), in Fourier space, we have :

$$V = \frac{c}{(s + a - \frac{b^2}{r}k)} Y_m \qquad (9)$$

introducing (9) in (8) we get :

$$Y = -\frac{1}{r} \frac{c^2 b^2}{(s - a + \frac{b^2}{r}k)(s + a - \frac{b^2}{r}k)} Y_m$$

and using (7) we can write :

$$Y = -\frac{1}{r} \frac{c^2 b^2}{(s - \sqrt{a^2 + \frac{b^2 c^2}{r}})(s + \sqrt{a^2 + \frac{b^2 c^2}{r}})} Y_m$$

$$Y = -\frac{1}{r} \frac{c^2 .b^2}{s^2 - (a^2 + \frac{b^2 .c^2}{r})} Y_m$$

and finally

$$Y = \frac{b^2 c^2}{b^2 c^2 - r.(s + a)(s - a)} Y_m \qquad (10)$$

*b-<u>The scalar case with the regularization method</u>*

The Fourier transform (1) is :

$$Y = \frac{bc}{s - a} U = H(s) \qquad (11)$$

The corresponding transfer function is given by :

$$\frac{H^*}{|H|^2 + r} = \frac{bc(s - a)}{b^2 c^2 - r(s + a)(s - a)} \qquad (12)$$

Introducing (11) in (12), the transfer function from $Y_m$ to $Y$ becomes :

$$Y = \frac{b^2c^2}{b^2c^2 - r(s+a)(s-a)} Y_m \tag{13}$$

It is straight forward to see that in both approaches (regularization method (13) and optimal control (10)) the transfer function from $Y$ to $Y_m$ is the same.

• The second step of the proof :

*a-*<u>*The matricial case with optimal control*</u>

Applying (2) to (1) the state space equation (1) becomes :

$$\begin{cases} \dot{x} = Ax - BR^{-1}B^t(Kx+v) \\ y = Cx \end{cases}$$

In Fourier space, we write :

$$(sI - (A - BR^{-1}B^tK))X = BR^{-1}B^tV$$

Where $X$, $K$ and $V$ are the Fourier transforms of $x$, $k$ and $v$, then we obtain with(1)

$$Y = -C(sI - (A - BR^{-1}B^tK))^{-1}BR^{-1}B^tV \tag{14}$$

using (4), in Fourier space

$$V = (sI + (A^t - KBR^{-1}B^t))^{-1}C^tQ.Y_m$$

(14) becomes

$$Y = -C(sI - (A - BR^{-1}B^tK))^{-1}BR^{-1}B^t(sI + (A^t - KBR^{-1}B^t))^{-1}C^tQ.Y_m \tag{15}$$

Developing the following term

$$\phi = (sI - (A - BR^{-1}B^tK))BR^{-1}B^t(sI + (A^t - KBR^{-1}B^t))^{-1}$$

with $BR^{-1}B^t=E$ we find

$$\phi = [sI - (A - EK)]^{-1}E.[sI + (A^t - KE)]^{-1}$$

$$\phi = [(sI - A) + EK]^{-1} E.[(sI + A^t) - KE]^{-1}$$

then it may be written as:

$$\phi = ([(sI + A^t) - KE]E^{-1}.[(sI - A) + EK])^{-1}$$

$$\phi = ([(sI + A^t) - KE][E^{-1}.(sI - A) + K])^{-1}$$

$$\phi = ((sI + A^t)E^{-1}.(sI - A) + (sI + A^t)K - K(sI - A) - KEK)^{-1}$$

using (3) and with $BR^{-1}B^t=E$

$$KA + A^t K + C^t QC = KEK$$

(16) becomes

$$\phi = ((sI + A^t)E^{-1}.(sI - A) - C^t QC)^{-1}$$

and we can write :

$$\phi = (((sI + A^t)E^{-1} - C^t QC(sI - A)^{-1})(sI - A))^{-1}$$

then

$$\phi = (((sI + A^t)E^{-1}BB^t(sI + A^t)^{-1} - C^t QC(sI - A)^{-1}BB^t(sI + A^t)^{-1})(sI + A^t)(BB^t)^{-1}(sI - A))^{-1}$$

where

$$E^{-1}BB^t = (BR^{-1}B^t)^{-1}BB^t = R(BB^t)^{-1}BB^t = R$$

It becomes

$$\phi = -(sI - A)^{-1}BB^t(sI + A^t)^{-1}(C^t QC(sI - A)^{-1}BB^t(sI + A^t)^{-1} - R.I)^{-1} \qquad (17)$$

Then we obtain (15) as follows

$$Y = -C(sI - A)^{-1} BB^t (sI + A^t)^{-1} (C^t QC(sI - A)^{-1} BB^t (sI + A^t)^{-1} - R.I)^{-1} C^t Q.Y_m$$

If we put

$$G = C(sI - A)^{-1} BB^t (sI + A^t)^{-1}$$

we can write the transfer between $Y_m$ and $Y$ as follows :

$$Y = G(C^t G - R.I)^{-1} C^t Y_m \tag{18}$$

here $Q=I$ ($I$ denotes the identity matrix)

*b-<u>The matricial case with regularization method</u>*

The Fourier transform (1) is :

$$H(s) = C(sI - A)^{-1} B$$

and

$$(C(-sI - A)^{-1} B)^t = -B^t ((sI + A)^t)^{-1} C^t = -B^t (sI + A^t)^{-1} C^t$$

we can write :

$$Y = B^t (sI + A^t)^{-1} C^t (C(sI - A)^{-1} BB^t (sI + A^t)^{-1} C^t - R)^{-1} U$$

$U$ represents the Fourier transform of $u$. We write the transfer between $Y$ and $Y_m$ under the form :

$$Y = C(sI - A)^{-1} BB^t (sI + A^t)^{-1} C^t (C(sI - A)^{-1} BB^t (sI + A^t)^{-1} C^t - R)^{-1} Y_m$$

Considering

$$C(sI - A)^{-1} BB^t (sI + A^t)^{-1} = G$$

we can write :

$$Y = GC^t (GC^t - R)^{-1} Y_m$$

We can verify that $(GC^t - R)$ is a scalar, consequently we can write :

$$Y = G(GC^t - R)^{-1} C^t Y_m \tag{19}$$

Now, the problem is to prove the equality between (18) and (19).

Let :

$$\psi = G(GC^t - R)^{-1}C^t = G(GC^t - R)^{-1}(C^tG - R.I)(C^tG - R.I)^{-1}C^t$$

$$\psi = G(GC^t - R)^{-1}C^tG(C^tG - R.I)^{-1}C^t - G(GC^t - R)^{-1}R(C^tG - R.I)^{-1}C^t$$

$$\psi = G(GC^t - R)^{-1}C^t = G(C^tG - R.I)^{-1}C^t$$

Since the term *(GC*$^t$ *- R)* is a scalar then

$$\psi = (\frac{GC^t}{GC^t - R} - \frac{R}{GC^t - R})G(C^tG - R.I)^{-1}C^t$$

*Remark 3*

From definition of $J_2$ and $J_3$, it is straightforward to see that considering $Q=I_{nxn}$ and $R=\alpha_0^2$ in $J_3$ yields $J_3=J_2$. Consequently, the duality of both methods in this context could be intuitively expected.

*5. Examples*

In the present section two examples are given in order to illustrate the equivalence between the first approach (regularization case) and the second approach (optimal control case).

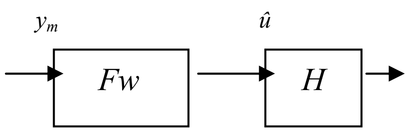


**Figure 3.1. Regularization technique.**

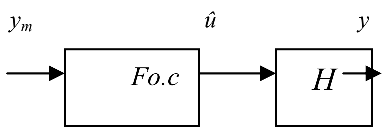


**Figure 3.2. Optimal control.**

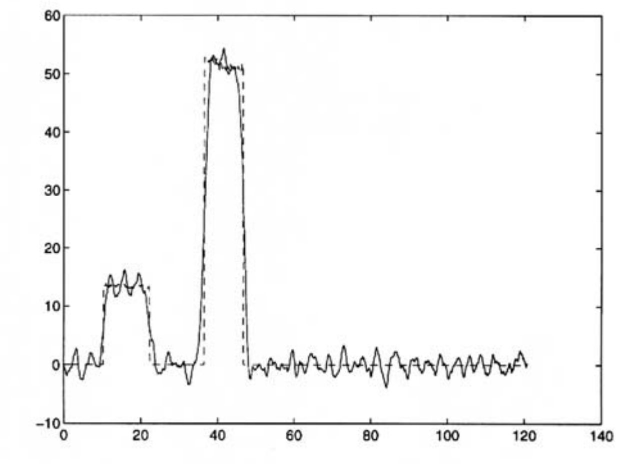

**Figure 4.1. Restoration with regularization technique.**

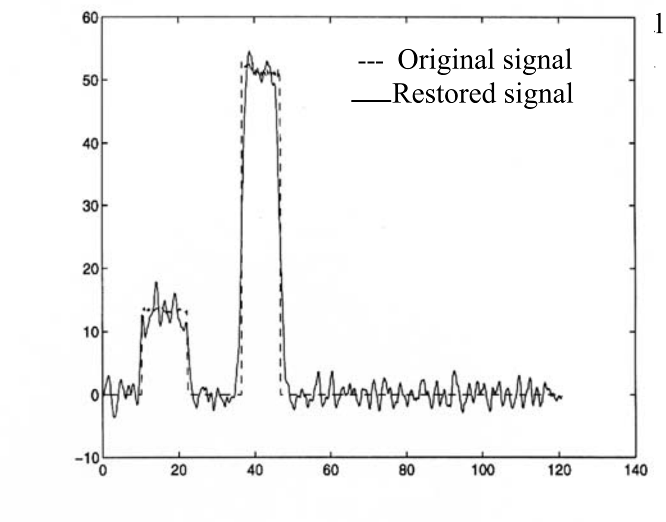


**Figure 4.2: Restoration with optimal control ($r$ =10$^{-4}$)**

## CONCLUSION

In the signal processing domain, the deconvolution problem is usually tackled through probabilistic consideration or through regularization. Recent works related to deconvolution have been developed using optimal control theory results. Considering a particular case of the latter approach, we can intuitively consider that regularization and non-stationary optimal control based on deconvolution are equivalent. In the present paper, the intuitive idea is proved to be mathematically true. The developed examples illustrate clearly the equality of both methods.

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